INTERACTIONS OF LANGUAGE AND MATHEMATICS: CESARO – LIKE SUMMATION FOR CERTAIN DIVERGENT SERIES

INTERAÇÕES DA LINGUAGEM E DA MATEMÁTICA: CESARO – COMO SOMATÓRIA PARA CERTAS SÉRIES DIVERGENTES

INTERACCIONES DEL LENGUAJE Y LAS MATEMÁTICAS: CESARO – COMO SUMA DE CIERTAS SERIES DIVERGENTES

Ramaswamy SIVARAMAN[1]
e-mail: rsivaraman1729@yahoo.co.in

How to reference this article:

SIVARAMAN, R. Interactions of language and mathematics: Cesaro – like summation for certain divergent series. **Rev. EntreLinguas**, Araraquara, v. 9, n. 00, e023007, 2023. e-ISSN: 2447-3529. DOI: https://doi.org/10.29051/el.v9i00.17895

| **Submitted**: 20/11/2022
| **Required revisions**: 25/12/2022
| **Approved**: 19/01/2023
| **Published**: 22/03/2023

Editor: Profa. Dra. Rosangela Sanches da Silveira Gileno
Deputy Executive Editor: Prof. Dr. José Anderson Santos Cruz

[1] Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai – India. Associate Professor, Department of Mathematics.

ABSTRACT: Mathematics has its special language including grammar and symbols shared by Mathematicians universally, regardless of their mother tongue. Since mathematics is the same all across the globe, math can serve as a global language. The idea of assigning certain specific finite values to given divergent series is called Cesaro Summation. This paper attempts to analyze the interaction between mathematics and language, considering Cesaro – like summation for certain divergent series. To meet that aim, Eulerian polynomials are utilized. Also, a novel method of determining Cesaro – Like summation values by integrating particular generating functions is defined over the closed and bounded intervals for a general infinite power series whose coefficients are *m*th powers of natural numbers. The answers obtained provide new insights into understanding the Cesaro – Like summation process and offer a great deal of generalization and also reveal the mysterious interaction of mathematics and language.

KEYWORDS: Language. Mathematics. Eulerian polynomials. Recurrence relation. Cesaro – like summation.

RESUMO*: A matemática tem sua linguagem especial, incluindo gramática e símbolos compartilhados por matemáticos universalmente, independentemente de sua língua materna. Como a matemática é a mesma em todo o mundo, ela pode servir como uma linguagem global. A ideia de atribuir certos valores finitos específicos a determinadas séries divergentes é chamada de Soma de Cesaro. Este artigo tenta analisar a interação entre matemática e linguagem, considerando Cesaro – como soma para certas séries divergentes. Para atingir esse objetivo, polinômios eulerianos são utilizados. Além disso, um novo método de determinação de valores de soma de Cesaro integrando funções geradoras particulares é definido sobre os intervalos fechados e limitados para uma série de potência infinita geral cujos coeficientes são milésimas potências de números naturais. As respostas obtidas fornecem novos insights sobre a compreensão do processo de soma de Cesaro e oferecem uma grande generalização e também revelam a misteriosa interação da matemática e da linguagem.*

PALAVRAS-CHAVE*: Linguagem. Matemática. Polinômios eulerianos. Relação de recorrência. Cesaro – como soma.*

RESUMEN*: Las matemáticas tienen su lenguaje especial, que incluye la gramática y los símbolos compartidos por los matemáticos universalmente, independientemente de su lengua materna. Como las matemáticas son las mismas en todo el mundo, pueden servir como un lenguaje global. La idea de asignar ciertos valores finitos específicos a ciertas series divergentes se llama Cesaro Sum. Este artículo trata de analizar la interacción entre matemática y lenguaje, considerando a Cesaro – como suma de ciertas series divergentes. Para lograr este objetivo, se utilizan polinomios eulerianos. Además, se define un nuevo método para determinar los valores de la suma de Cesaro integrando funciones generadoras particulares sobre los intervalos cerrados y acotados para una serie general de potencias infinitas cuyos coeficientes son m-ésimas potencias de números naturales. Las respuestas obtenidas brindan nuevos conocimientos para comprender el proceso de suma de Cesaro y ofrecen una gran generalización y también revelan la misteriosa interacción de las matemáticas y el lenguaje.*

PALABRAS CLAVE*: Lenguaje. Matemáticas. Polinomios eulerianos. Relación de recurrencia. Cesaro – como suma.*

Introduction

The mathematical language is regarded as a huge extension of the natural language utilized in science and mathematics for stating outcomes with precision, concision and unambiguity (RIMM-KAUFMAN *et al.*, 2015; HOFMANN; MERCER, 2016; PURPURA; REID, 2016).

Language and mathematics do not appear to be subjects as separate as one may imagine them to be. Mathematics has its peculiar notation or language including symbols unique to mathematics, like the '=' symbol (LEHRL *et al.*, 2020; PURPURA; REID, 2016; MARTIN; RIMM-KAUFMAN, 2015).

The concept of assigning certain values for infinite divergent series was attributed to Italian mathematician Ernesto Cesaro (BLUMS *et al.*, 2017; ULATOWSKI *et al.*, 2016; GENLOTT; GRÖNLUND, 2016). The Cesaro summation discuss about the limit of sequence of partial sums of a given series (LEYVA *et al.*, 2015). Ever since this idea emerged, several mathematicians generalized it in various forms (REDISH; KUO, 2015; PENG *et al.*, 2020). In this paper, I will determine the Cesaro – Like Summation for two specific infinite divergent series related to sum of powers using suitable generating functions. We encounter Eulerian polynomials for performing such task. Some graphs are displayed to verify the results obtained.

Methods

Eulerian Polynomials

Eulerian polynomials are class of polynomials whose coefficients occur in counting number of permutations with particular descents. The first few Eulerian polynomials are given by

$$\left.\begin{array}{l} E_0(x)=1, E_1(x)=1+x, E_2(x)=1+4x+x^2, E_3(x)=1+11x+11x^2+x^3, \\ E_4(x)=1+26x+66x^2+26x^3+x^4, E_5(x)=1+57x+302x^2+302x^3+57x^4+x^5,\ldots \end{array}\right\} \quad (1)$$

We notice that the coefficients of Eulerian polynomials which are called Eulerian Numbers are symmetric and hence $E_n(-1)=0$ whenever n is an odd positive integer.

Generating functions

First, we notice the following generating functions for infinite series whose coefficients are related to *m*th powers of natural numbers.

If *m* is a positive integer, let us assume that

$$S_m(x) = 1^m + 2^m x + 3^m x^2 + 4^m x^3 + 5^m x^4 + \cdots \tag{2}$$

For any real number *x*, we notice from (2), that $S_m(x)$ is an infinite power series in *x* which is divergent for all *x* such that $|x| \geq 1$. Moreover, the coefficients of $S_m(x)$ are *m*th powers of natural numbers.

From (1) and (2), we obtain the following equations:

$$\left.\begin{aligned}
\frac{E_0(x)}{(1-x)^2} &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = S_1(x) \\
\frac{E_1(x)}{(1-x)^3} &= \frac{1+x}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + 5^2 x^4 + \cdots = S_2(x) \\
\frac{E_2(x)}{(1-x)^4} &= \frac{1+4x+x^2}{(1-x)^4} = 1 + 2^3 x + 3^3 x^2 + 4^3 x^3 + 5^3 x^4 + \cdots = S_3(x) \\
\frac{E_3(x)}{(1-x)^5} &= \frac{1+11x+11x^2+x^3}{(1-x)^5} = 1 + 2^4 x + 3^4 x^2 + 4^4 x^3 + 5^4 x^4 + \cdots = S_4(x) \\
\frac{E_4(x)}{(1-x)^6} &= \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6} = 1 + 2^5 x + 3^5 x^2 + 4^5 x^3 + 5^5 x^4 + \cdots = S_5(x) \\
&\;\;\cdots
\end{aligned}\right\} \tag{3}$$

In general, for any positive integer *m*, we have

$$\frac{E_{m-1}(x)}{(1-x)^{m+1}} = 1^m + 2^m x + 3^m x^2 + 4^m x^3 + 5^m x^4 + \cdots = S_m(x) \tag{4}$$

Thus $\frac{E_{m-1}(x)}{(1-x)^{m+1}}$ behaves as the generating functions for each *m* for the power series $S_m(x)$.

Cesaro – like summation

The Cesaro – Like Summation for the power series $S_m(x)$ at $x = k$ is defined as

$$(CL)\left(1^m + 2^m k + 3^m k^2 + 4^m k^3 + 5^m k^4 + \cdots\right) = (CL)\left(S_m(k)\right) = \int_{x=-k}^{0} \frac{E_{m-1}(x)}{(1-x)^{m+1}} dx \tag{5}$$

where k is any positive real number. Here CL refers to Cesaro – Like summation process.

We now prove a theorem related to finding Cesaro – Like summation for the power series $S_m(x)$.

Theorem 1

The Cesaro – Like summation of the power series $S_m(x) = 1^m + 2^m x + 3^m x^2 + 4^m x^3 + 5^m x^4 + \cdots$ for $m \geq 2$ at $x = k$ is given by

$$(CL)\left(S_m(k)\right) = \frac{kE_{m-2}(-k)}{(k+1)^m} \tag{6}$$

where $E_{m-2}(-k)$ are Eulerian polynomials evaluated at $x = -k$ and k is any positive real number.

Proof: Using (6), we get $(CL)\left(S_m(k)\right) = \int_{x=-k}^{0} \frac{E_{m-1}(x)}{(1-x)^{m+1}} dx$

We notice that the function $\frac{E_{m-1}(x)}{(1-x)^{m+1}}$ is continuous in $[-k, 0]$, Hence the integral defined above exists. We know that the Eulerian polynomials satisfy the recurrence relation

$$E_m(x) = x(1-x)E'_{m-1}(x) + (mx+1)E_{m-1}(x) \tag{7}$$

Now using (7) and Integration by parts formula, we obtain

$$\begin{aligned}(CL)\left(S_m(k)\right) &= \int_{x=-k}^{0} \frac{E_{m-1}(x)}{(1-x)^{m+1}} dx = \int_{x=-k}^{0} \frac{\left[x(1-x)E'_{m-2}(x) + ((m-1)x+1)E_{m-2}(x)\right]}{(1-x)^{m+1}} dx \\ &= \int_{x=-k}^{0} \frac{xE'_{m-2}(x)}{(1-x)^m} dx + \int_{x=-k}^{0} \frac{\left[((m-1)x+1)E_{m-2}(x)\right]}{(1-x)^{m+1}} dx \\ &= \left[\left(\frac{x}{(1-x)^m}\right)E_{m-2}(x)\right]_{x=-k}^{0} - \int_{x=-k}^{0} \frac{\left[((m-1)x+1)E_{m-2}(x)\right]}{(1-x)^{m+1}} dx + \int_{x=-k}^{0} \frac{\left[((m-1)x+1)E_{m-2}(x)\right]}{(1-x)^{m+1}} dx \\ &= \frac{kE_{m-2}(-k)}{(k+1)^m}\end{aligned} \tag{8}$$

This completes the proof.

Corollary

If m is a positive integer, then $(CL)\left(S_{2m+1}(1)\right)=0$

Proof: Using (1-8) we get

$$(CL)\left(S_{2m+1}(1)\right)=\int_{x=-1}^{0}\frac{E_{2m}(x)}{(1-x)^{2m+2}}dx=\left[\frac{kE_{2m-1}(-k)}{(k+1)^{2m+1}}\right]_{k=1}=\frac{E_{2m-1}(-1)}{2^{2m+1}}=0 \tag{9}$$

This completes the proof.

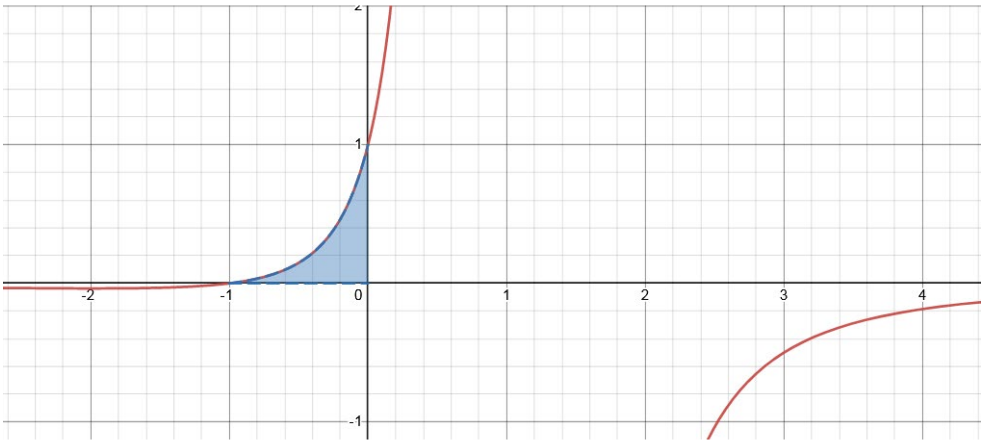
From (9), we note that

$$\left.\begin{array}{l}(CL)(1^3+2^3+3^3+4^3+5^3+\cdots)=0,(CL)(1^5+2^5+3^5+4^5+5^5+\cdots)=0,\\(CL)(1^7+2^7+3^7+4^7+5^7+\cdots)=0,(CL)(1^9+2^9+3^9+4^9+5^9+\cdots)=0,\ldots\end{array}\right\} \tag{10}$$

Results and discussion

In this section, we try to verify the results obtained in section 4 using graphs of generating functions of $S_m(x)$ at $x = k$, where k is a positive real number.

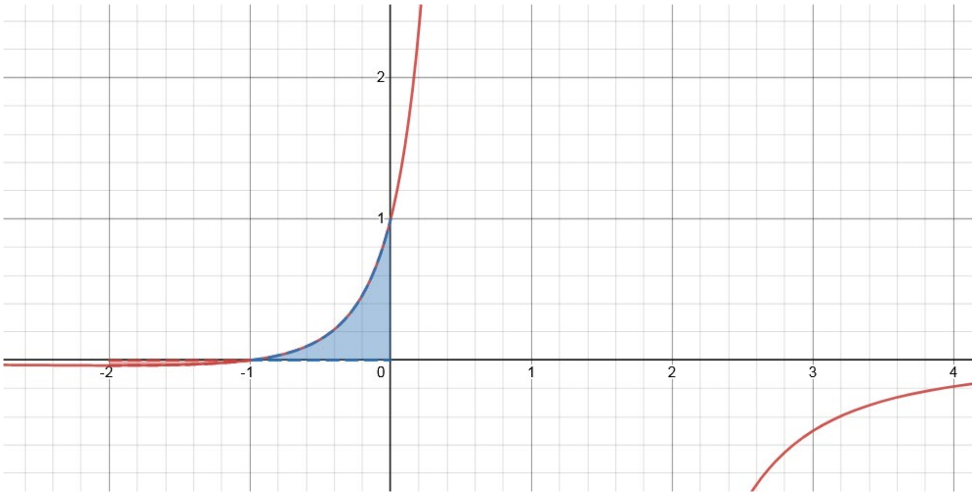
Figure 1: $\int_{-1}^{0}\frac{1+x}{(1-x)^3}dx=\int_{-1}^{0}S_2(x)dx=\frac{1}{4}$



Source: Devised by the author

If we consider the interval [–1, 0] then with k = 1, m = 2 we obtain $(CL)\left(S_2(1)\right)=(CL)(1^2+2^2+3^2+4^2+5^2+\cdots)=\int_{x=-1}^{0}S_2(x)dx=\frac{E_0(-1)}{2^2}=\frac{1}{4}$. This verifies the calculation shown in Figure 1.

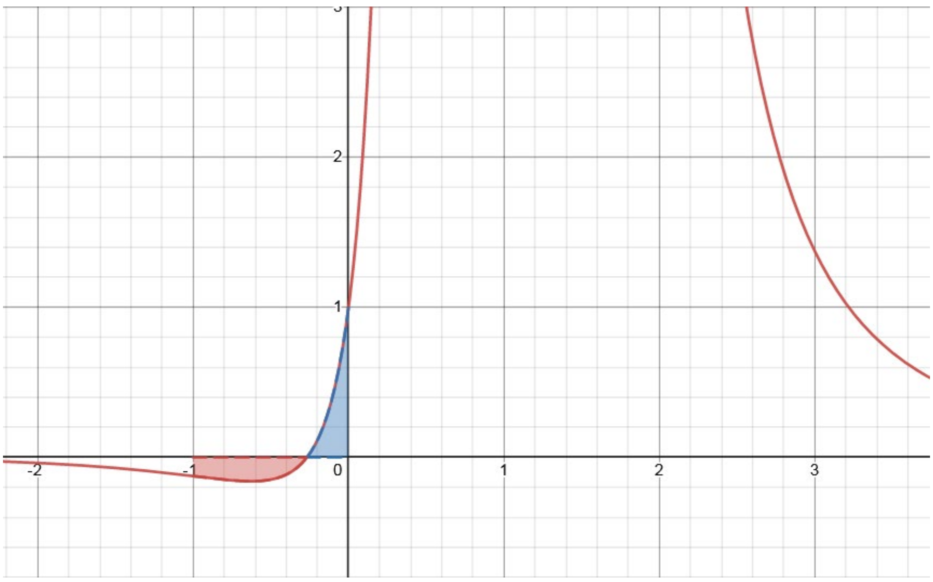
Figure 2: $\int_{-2}^{0} \frac{1+x}{(1-x)^3} dx = \int_{-2}^{0} S_2(x) dx = \frac{2}{9}$



Source: Devised by the author

If we consider the interval [–2, 0] then, with $k = 2$, $m = 2$ we obtain $(CL)\left(S_2(2)\right) = (CL)(1^2 + 2^2 \times 2 + 3^2 \times 2^2 + 4^2 \times 2^3 + 5^2 \times 2^4 + \cdots) = \int_{x=-2}^{0} S_2(x) dx = \frac{2E_0(-2)}{3^2} = \frac{2}{9}$.
This verifies the calculation shown in Figure 2.

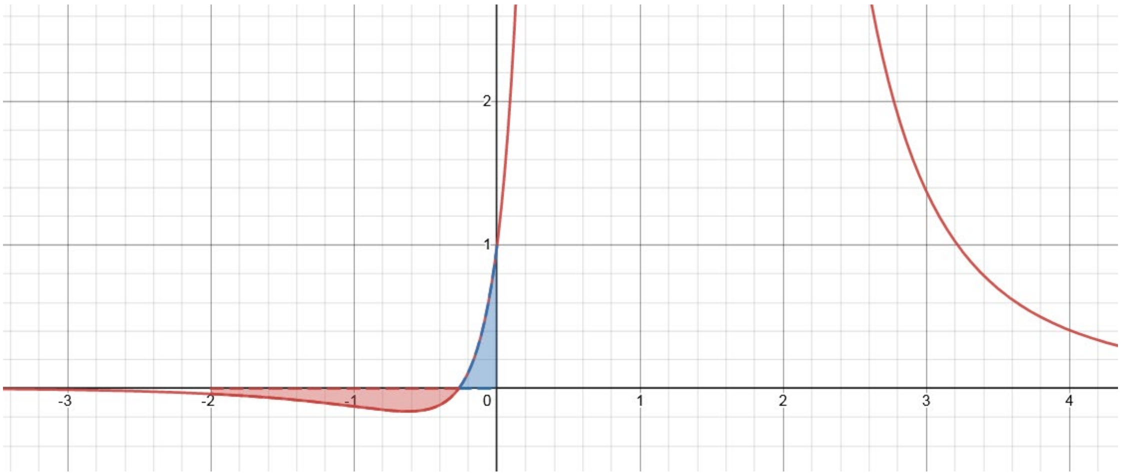
Figure 3: $\int_{-1}^{0} \frac{1+4x+x^2}{(1-x)^4} dx = \int_{-1}^{0} S_3(x) dx = 0$



Source: Devised by the author

If we consider the interval [–1, 0] then with k = 1, m = 3 we obtain $(CL)\left(S_3(1)\right) = (CL)(1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots) = \int_{x=-1}^{0} S_3(x)dx = \frac{E_1(-1)}{2^3} = 0$. This verifies the calculation shown in Figure 3.

Figure 4: $\int_{-2}^{0} \frac{1+4x+x^2}{(1-x)^4} dx = \int_{-2}^{0} S_3(x)dx = -\frac{2}{27}$

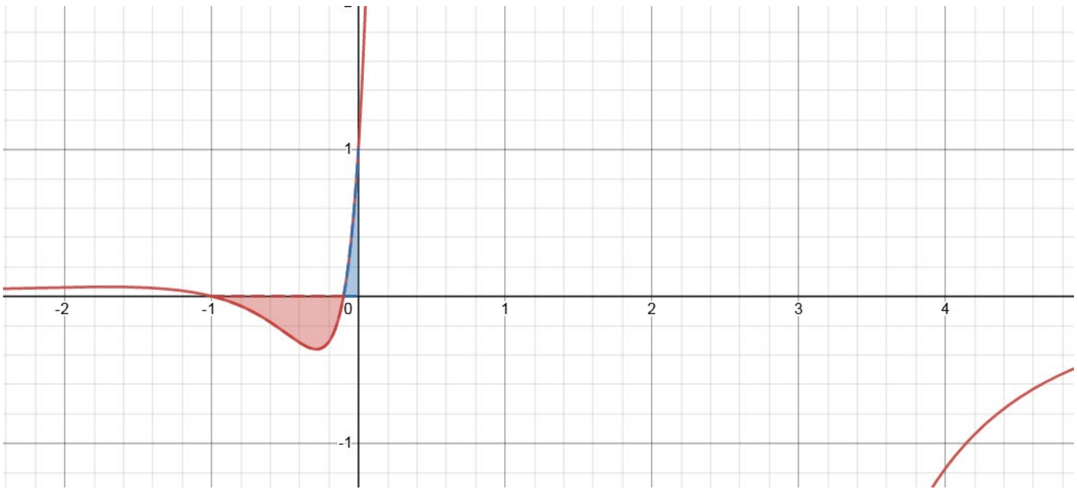


Source: Devised by the author

If we consider the interval [–2, 0] then with k = 2, m = 3 we get $(CL)\left(S_3(2)\right) = (CL)(1^3 + 2^3 \times 2 + 3^3 \times 2^2 + 4^3 \times 2^3 + 5^3 \times 2^4 + \cdots) = \int_{x=-2}^{0} S_3(x)dx = \frac{2E_1(-2)}{3^3} = -\frac{2}{27}$

This verifies the calculation shown in Figure 4.

Figure 5: $\int_{-1}^{0} \frac{1+11x+11x^2+x^3}{(1-x)^5} dx = \int_{-1}^{0} S_4(x)dx = -\frac{1}{8}$



Source: Devised by the author

If we consider the interval [–1, 0] then with k = 1, m = 4 we obtain $(CL)\left(S_4(1)\right) = (CL)(1^4 + 2^4 + 3^4 + 4^4 + 5^4 + \cdots) = \int_{x=-1}^{0} S_4(x)dx = \frac{E_2(-1)}{2^4} = -\frac{1}{8}$. This verifies the calculation shown in Figure 5.

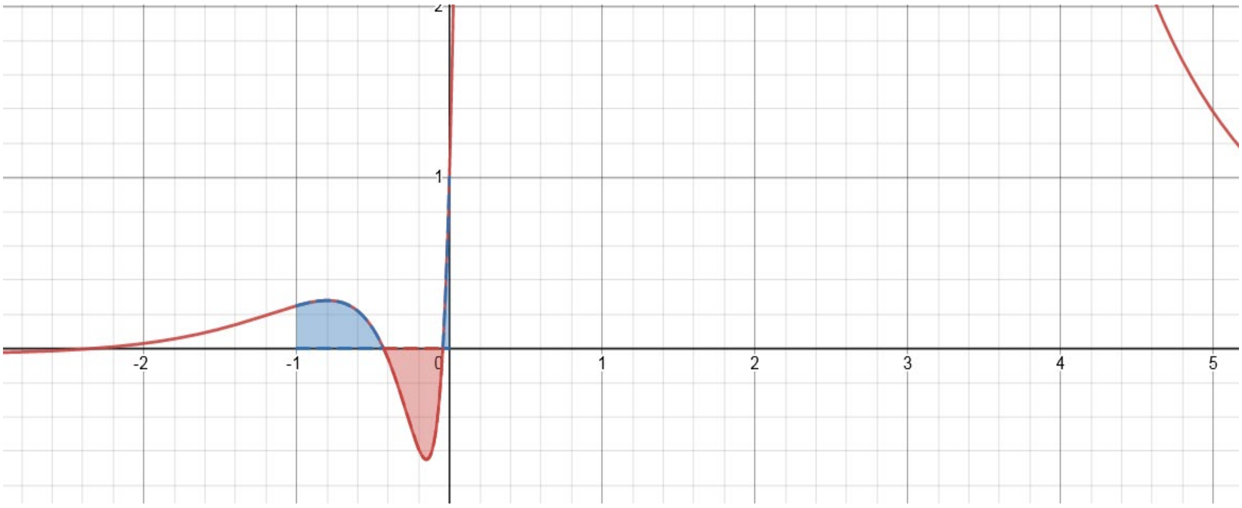
Figure 6: $\int_{-2}^{0} \frac{1+11x+11x^2+x^3}{(1-x)^5}dx = \int_{-2}^{0} S_4(x)dx = -\frac{2}{27}$

Source: Devised by the author

If we consider the interval [–2, 0] then with k = 2, m = 4 we get $(CL)\left(S_4(2)\right) = (CL)(1^4 + 2^4 \times 2 + 3^4 \times 2^2 + 4^4 \times 2^3 + 5^4 \times 2^4 + \cdots) = \int_{x=-2}^{0} S_4(x)dx = \frac{2E_2(-2)}{3^4} = -\frac{2}{27}$

This verifies the calculation shown in Figure 6.

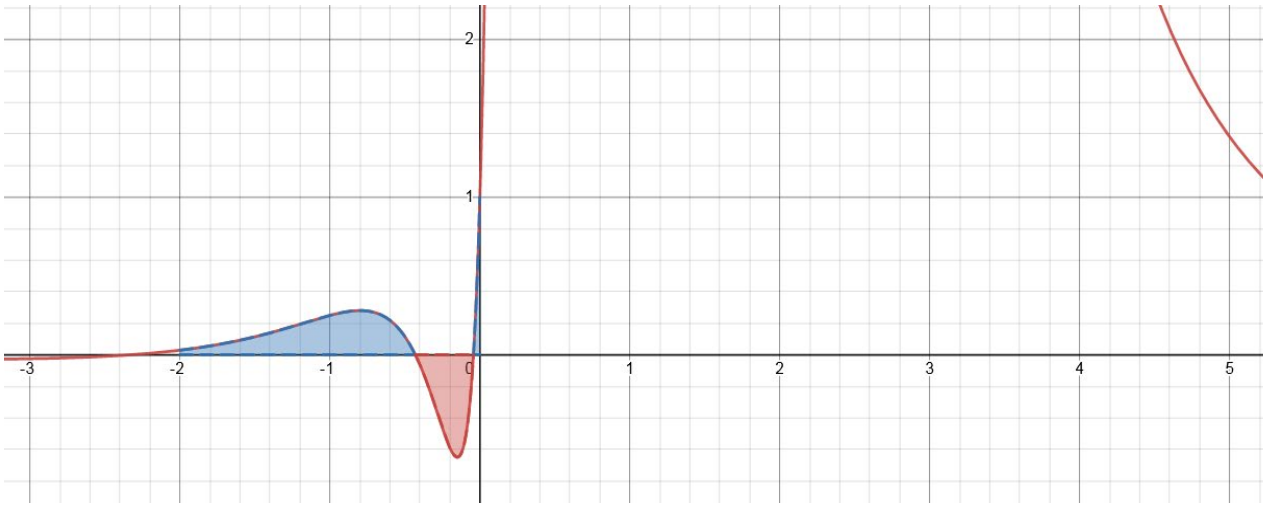
Figure 7: $\int_{-1}^{0} \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6} dx = \int_{-1}^{0} S_5(x)dx = 0$



Source: Devised by the author

If we consider the interval [–1, 0] then with k = 1, m = 5 we obtain $(CL)\left(S_5(1)\right) = (CL)(1^5+2^5+3^5+4^5+5^5+\cdots) = \int_{x=-1}^{0} S_5(x)dx = \frac{E_3(-1)}{2^5} = 0$. This verifies the calculation shown in Figure 7.

Figure 8: $\int_{-2}^{0} \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6} dx = \int_{-2}^{0} S_5(x)dx = \frac{10}{81}$

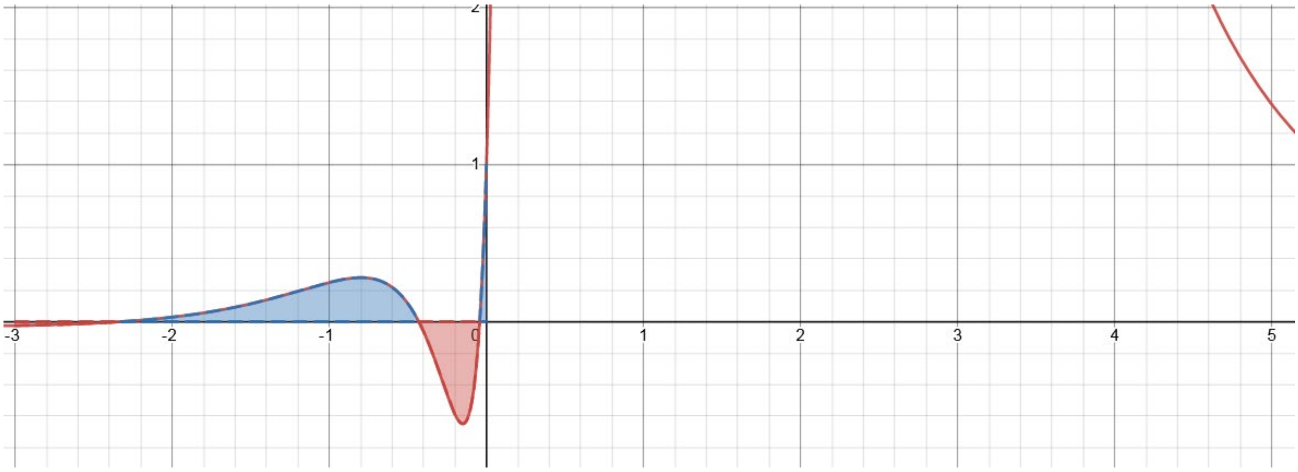


Source: Devised by the author

If we consider the interval [–2, 0] then with k = 2, m = 5 we get $(CL)\left(S_5(2)\right) = (CL)(1^5 + 2^5 \times 2 + 3^5 \times 2^2 + 4^5 \times 2^3 + 5^5 \times 2^4 + \cdots) = \int_{x=-2}^{0} S_5(x)dx = \frac{2E_3(-2)}{3^5} = \frac{10}{81}$

This verifies the calculation shown in Figure 8.

Figure 9: $\int_{-3}^{0} \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6}dx = \int_{-3}^{0} S_5(x)dx = \frac{30}{256}$

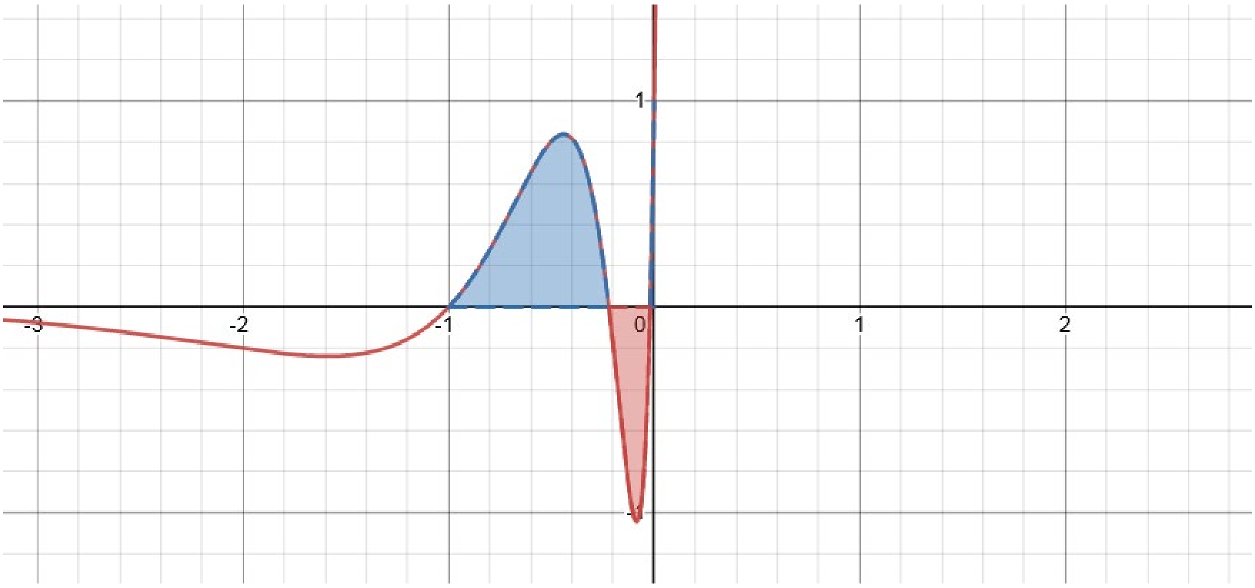


Source: Devised by the author

If we consider the interval [–3, 0] then with k = 3, m = 5 we get $(CL)\left(S_5(3)\right) = (CL)(1^5 + 2^5 \times 3 + 3^5 \times 3^2 + 4^5 \times 3^3 + 5^5 \times 3^4 + \cdots) = \int_{x=-3}^{0} S_5(x)dx = \frac{3E_3(-3)}{4^5} = \frac{30}{256}$.

This verifies the calculation shown in Figure 9.

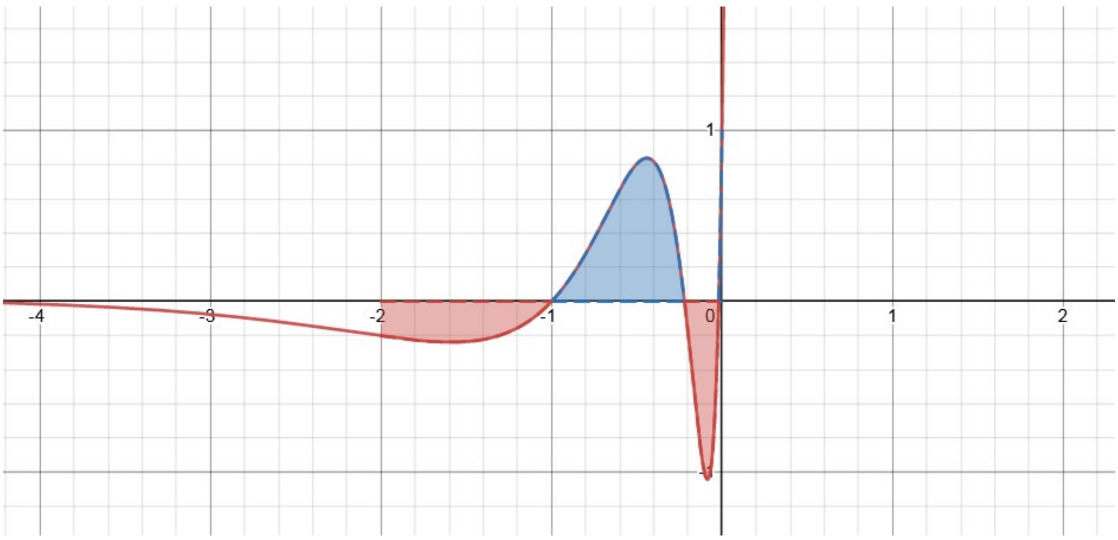
Figure 10: $\displaystyle\int_{-1}^{0} \frac{1+57x+302x^2+302x^3+57x^4+x^5}{(1-x)^7}dx = \int_{-1}^{0} S_6(x)dx = \frac{1}{4}$



Source: Devised by the author

If we consider the interval [–1, 0] then with k = 1, m = 6 we obtain $(CL)\big(S_6(1)\big) = (CL)(1^6+2^6+3^6+4^6+5^6+\cdots) = \int_{x=-1}^{0} S_6(x)dx = \frac{E_4(-1)}{2^6} = \frac{1}{4}$. This verifies the calculation shown in Figure 10.

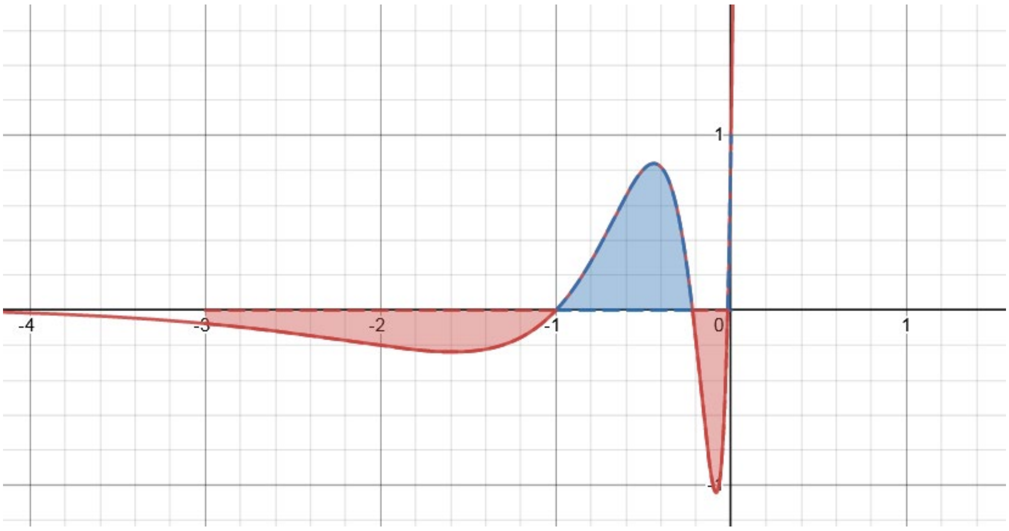
Figure 11: $\displaystyle\int_{-2}^{0} \frac{1+57x+302x^2+302x^3+57x^4+x^5}{(1-x)^7}dx = \int_{-2}^{0} S_6(x)dx = \frac{14}{243}$



Source: Devised by the author

If we consider the interval [–2, 0] then $k = 2$, $m = 6$ we get $(CL)\left(S_6(2)\right) = (CL)(1^6 + 2^6 \times 2 + 3^6 \times 2^2 + 4^6 \times 2^3 + 5^6 \times 2^4 + \cdots) = \int_{x=-2}^{0} S_6(x)dx = \frac{2E_4(-2)}{3^6} = \frac{14}{243}$.
This verifies the calculation shown in Figure 11.

Figure 12: $\int_{-3}^{0} \frac{1+57x+302x^2+302x^3+57x^4+x^5}{(1-x)^7}dx = \int_{-3}^{0} S_6(x)dx = -\frac{39}{512}$



Source: Devised by the author

If we consider the interval [–3, 0] then with $k = 3$, $m = 6$ we get $(CL)\left(S_6(3)\right) = (CL)(1^6 + 2^6 \times 3 + 3^6 \times 3^2 + 4^6 \times 3^3 + 5^6 \times 3^4 + \cdots) = \int_{x=-3}^{0} S_6(x)dx = \frac{3E_4(-3)}{4^6} = -\frac{39}{512}$

This verifies the calculation shown in Figure 12.

Conclusion

By considering the idea of Cesaro – Like summation for a particular power series $S_m(x)$ I had obtained a nice, closed expression for computing $(CL)\left(S_m(k)\right)$ for any positive real number *k*. The answer given by Theorem 1 of this paper, is a rational expression obtained as function of *k*. Hence choosing convenience positive values of *k*, through this rational expression, we can generate as many Cesaro – Like summation values as we wish.

Curiously, for the summation process described in this paper, we notice that Cesaro – Like summation values of sum of odd powers of natural numbers are always zero. Twelve figures were displayed using Desmos free online software for computing definite integrals to verify the formula obtained in theorem 1. Hence, the ideas presented in this paper, provide us the way for obtaining several Cesaro – Like summation values for each positive k and for positive integer m such that $m \geq 2$. We can generalize the way we have defined the Cesaro – Like summation as in and try to obtain other interesting results.

REFERENCES

BLUMS, A. *et al*. Building links between early socioeconomic status, cognitive ability, and math and science achievement. **Journal of Cognition and Development**, v. 18, n. 1, p. 16-40, 2017.

GENLOTT, A. A.; GRÖNLUND, Å. Closing the gaps–Improving literacy and mathematics by ict-enhanced collaboration. **Computers & Education**, n. 99, p. 68-80, 2016.

HOFMANN, R.; MERCER, N. Teacher interventions in small group work in secondary mathematics and science lessons. **Language and education**, v. 30, n. 5, p. 400-416, 2016.

LEHRL, S. *et al*. Long-term and domain-specific relations between the early years home learning environment and students' academic outcomes in secondary school. **School Effectiveness and School Improvement**, v. 31, n. 1, p. 102-124, 2020.

LEYVA, D. *et al*. Teacher–child interactions in Chile and their associations with prekindergarten outcomes. **Child development**, v. 86, n. 3, p. 781-799, 2015.

MARTIN, D. P.; RIMM-KAUFMAN, S. E. Do student self-efficacy and teacher-student interaction quality contribute to emotional and social engagement in fifth grade math?. **Journal of school psychology**, v. 53, n. 5, p. 359-373, 2015.

PENG, P. *et al*. Examining the mutual relations between language and mathematics: A meta-analysis. **Psychological Bulletin**, v. 146, n. 7, p. 595, 2020.

PURPURA, D. J.; REID, E. E. Mathematics and language: Individual and group differences in mathematical language skills in young children. **Early Childhood Research Quarterly**, v. 36, p. 259-268, 2016.

REDISH, E. F.; KUO, E. Language of physics, language of math: Disciplinary culture and dynamic epistemology. **Science & Education**, v. 24, n. 5, p. 561-590, 2015.

RIMM-KAUFMAN, S. E. *et al*. To what extent do teacher–student interaction quality and student gender contribute to fifth graders' engagement in mathematics learning?. **Journal of Educational Psychology**, v. 107, n. 1, p. 170, 2015.

ULATOWSKI, F. *et al*. Recognizing the limited applicability of Job plots in studying host–guest interactions in supramolecular chemistry. **The Journal of organic chemistry**, v. 81, n. 5, p. 1746-1756, 2016.

Processing and editing: Editora Ibero-Americana de Educação.
Proofreading, formatting, normalization and translation.