SPECIFICITIES OF TEACHER'S INTERPRETATIVE KNOWLEDGE AND TASKS FOR TEACHER EDUCATION AS ELEMENTS FOR CREATIVE AND INNOVATIVE MATHEMATICAL PRACTICES

ESPECIFICIDADES DO CONHECIMENTO INTERPRETATIVO DO PROFESSOR E DAS TAREFAS PARA A FORMAÇÃO COMO ELEMENTOS PARA PRÁTICAS CRIATIVAS E MATEMATICAMENTE INOVADORAS

ESPECIFICIDADES DE LOS CONOCIMIENTOS INTERPRETATIVOS Y LAS TAREAS FORMATIVAS DEL DOCENTE COMO ELEMENTOS PARA PRÁCTICAS CREATIVAS Y MATEMÁTICAMENTE INNOVADORAS

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ABSTRACT: In the context of teacher education focusing on the specificities of teachers' specialized knowledge, it's evident the need for innovative and scientifically supported proposals for research with replicable approaches focusing on the foundational dimensions to improve the quality of discussions and students' mathematical learning. Considering the specificities of teacher's practices that enable students to understand, in order to be able to assume has a starting point the students' knowledge, a specialized knowledge that allows listening to students' mathematical thinking is required – called Interpretive Knowledge. This specialized knowledge is not developed during practice, requiring teacher education contexts with such a purpose. In this paper, we discuss innovation associated with theoretical and methodological research approaches, the conceptualization of Tasks for Teacher Education (within the scope of Isometric Geometric Transformations) and the methodological approach associated with its implementation in contexts intertwining teacher education and research.

KEYWORDS: Interpretive Knowledge. Tasks for Teacher Education. Isometric Geometric Transformations.

RESUMO*: No contexto da formação de professores com foco nas especificidades do conhecimento especializado do professor, é evidente a necessidade de uma formação inovadora e, cientificamente, sustentada para desenvolver pesquisas com abordagens replicáveis que foquem as dimensões fundamentais para melhorar a qualidade das discussões e das aprendizagens matemáticas dos alunos. Considerando as especificidades da prática profissional do professor que possibilitam o entendimento dos alunos, a partir do conhecimento que possuem, é requerido um conhecimento especializado que permita escutar o Pensar matemático dos alunos – denominado Conhecimento Interpretativo – e esse conhecimento não se desenvolve na prática de sala de aula, requerendo contextos formativos com esse fito. Neste artigo, discutimos inovação associada às abordagens teóricas e metodológicas de pesquisa, à conceitualização das Tarefas para a Formação especializada (no âmbito das Transformações Geométricas Isométricas) e à abordagem metodológica associada à sua implementação em contextos imbricando formação e pesquisa.*

PALAVRAS-CHAVE*: Conhecimento Interpretativo. Tarefas para a Formação. Transformações Geométricas Isométricas.*

RESUMEN*: En el contexto de la formación docente con un enfoque en las especificidades del conocimiento especializado del docente, se evidencia la necesidad de una formación innovadora y científicamente sustentada para desarrollar investigaciones con enfoques replicables que se centren en las dimensiones fundamentales para mejorar la calidad de las discusiones y el aprendizaje matemático de los estudiantes. Considerando las especificidades de la práctica profesional docente que posibilitan la comprensión de los estudiantes, a partir de los conocimientos que poseen, se requiere un conocimiento especializado que permita escuchar el pensamiento matemático de los estudiantes – llamado Conocimiento Interpretativo – y este conocimiento no se desarrolla en la práctica del aula, requiriendo contextos formativos con este propósito. En este artículo, discutimos la innovación asociada a los enfoques teóricos y metodológicos de la investigación, la conceptualización de las Tareas para la Formación Especializada (en el ámbito de las Transformaciones Geométricas Isométricas) y el enfoque metodológico asociado a su implementación en contextos que entrelazan la formación y la investigación.*

PALABRAS CLAVE*: Conocimientos interpretativos. Tareas para la formación. Transformaciones geométricas isométricas.*

Introduction

Thinking and carrying out innovation – in terms of results or processes – must be associated with doing something that has not yet been done, doing it differently than usual, or both. In the context of Mathematics Education that seeks to improve student results in mathematics, innovation cannot be understood as changing pedagogical approaches and continuing in the same space of mathematical discussion, but demands considering changing ways of thinking and developing mathematical knowledge of students as a priority. This innovation requires specialized and mathematically innovative professional practice.

The mathematics teacher's practice is based on tasks, preparing them and implementing them with students (Mason; Johnston-Wilder, 2006). Each type of task (Ponte, 2005), however, is associated with different objectives and a specific way of understanding the role of the teacher and students (Stein *et al*., 2000; Watson; Sullivan, 2008). Among the diversity of types and forms of tasks – introduction, consolidation, review, evaluation, involving exercises, problem solving, problem formulation, or investigations – our prioritization for thinking and carrying out innovation is directed towards tasks introduction of topics - as these are the moments in which the teacher mobilizes his knowledge in a more accessible way (Ribeiro, 2013; Ribeiro; Carrillo; Monteiro, 2012; Shoenfeld, 2000) - and associated with the resolution and formulation of problems (or investigations) because they are the contexts that lead students to have to think mathematically in a way they have never done before, or they would not be real problems.

It is also considered a parallelism between the teacher's practice with students and the trainer's formative practice, both in methodological terms of enabling the teacher to experience what is expected to be able, subsequently, to provide to their students (we assume that pedagogical knowledge it is not taught, it is experienced) as in mathematical terms, as the teacher has to start to understand mathematics and think mathematically so that it is possible, later, to propose tasks and carry out discussions that allow students to develop their ways of thinking mathematically, and this requires doing something different from what has been done, or this focus on training would not be necessary. To develop specialized training for specialized teachers, we consider the so-called Training Tasks – TpF (Ribeiro; Almeida; Mellone, 2021) as a specializing and specialized pedagogical resource, taking into account the way in which we assume our role as teachers, the role of students and the role of trainers. Each TpF is accompanied by a set of other three documents (which, together, constitute the Formative Tasks) that sustain and support specialized training that pursues the objectives of developing the teacher's specialized knowledge and transforming their mathematical practices into something pedagogically exciting and mathematically innovative, enabling students to enjoy learning, as they understand what they do and why they do it, at each moment and with future connections. This priority focus of training on the teacher's specialized knowledge considers the fact that this knowledge is, among the controllable factors, the one that most impacts students' learning and results (Baumert *et al*., 2010; Grossman, 2010; Nye; Konstantopoulos; Hedges, 2004).

Among the panoply of ways of considering teacher knowledge – from a perspective that focuses on generalities (Ribeiro, 2018) to one that conceives specificities, we assume the latter. In this sense, we seek to break with several of the assumptions established and implemented today in teacher training - generalities, such as that it is enough to have been a student in the educational stage to teach and replicate the experience to teach (absence of any discussion in training) information about the topics that will need to be taught); that it is enough to know how to do it (training of future teachers and future ordinary mathematicians) and with an instrumental character (Lopes *et al*., 2022); that, to improve results, it is enough to change the methodologies to the more "attractive" ones (training focusing on "fashionable" methodologies without mathematical discussion) – and we assume the teacher's knowledge as specialized from the perspective of the theoretical conceptualizations of *Mathematics Teacher's Specialized Knowledge* [3]– MTSK (Carrillo *et al*., 2018) and Interpretive Knowledge – CI (Di Martino; Mellone; Ribeiro, 2020; Jakobsen; Ribeiro; Mellone, 2014; Ribeiro; Mellone; Jakobsen, 2013).

We assume, in an associated way, a perspective of innovation also in methodological terms of implementing TpF in training and research contexts, assumed in an overlapping manner. The Individual-Collective-Individual – ICI (Pacelli *et al*., 2020) or Small group-Collective-Small group – Pg - C-Pg training cycles are considered as methodological approaches for implementing TpF (Jakobsen; Ribeiro; Mellone, 2022; Mellone *et al*., 2023), as a process that allows innovation in terms of results to be achieved through the focus of different types of individual-collective discussions.

In this text, we carry out a theoretical discussion based on examples of proposals for research and training, both specialized, not discussing the methodological approach to research that we developed, but focusing attention here on the dimensions of educational innovation that are considered. We discuss innovation in three dimensions: (i) theoretical (ways of understanding teacher knowledge); (ii) resources for collecting information (Training Tasks)

[3]We chose to use the nomenclature in English because it is already recognized internationally and the translation may result in a demeaning , which is associated with each of the dimensions of the conceptualization.

and developing the teacher's specialized knowledge (Training Tasks and Interpretive Tasks); and (iii) methodological approach to implementing Training Tasks and conceptualizing Training Tasks to maximize the quality of discussions and the sustainability of the development of the teacher's specialized knowledge. For this discussion, we bring an example of a Formative Task and the TpF associated with the rotation isometric transformation. This example serves as a generator of discussion and promoter of understanding, as experience shows that any innovation requires breaking with the chains that restrict us (Ribeiro, 2013) and doing what has not yet been done and, by presenting concrete examples that allow us to sustain The discussions are expected to lead the reader, starting from this specification, to reach a generalization of the ideas presented.

Some theoretical discussions

Students have difficulties in several mathematical topics (Clements; Sarama, 2020; Kieren, 1976; Ma, 1999) and, more generally, difficulties in Thinking and Thinking mathematically. Among the themes in which they reveal the greatest difficulties is Geometry and, within this, isometric geometric transformations assume a prominent place, not only because of the difficulties (see for example, Bairral; Silva, 2010; Gaspar; Cabrita, 2014; Küchemann, 1981), but through the connections that can (and should) be established with other mathematical and extra-mathematical themes and topics, in order to enhance the development of this Thinking mathematically in terms of understanding the mathematical structure and the elements that support demonstration and generalization.

Rotation is one of the three isometric geometric transformations (the others are reflection and translation) and because it is isometric it preserves distances (Lima, 1992) and range of angles, which leads to congruence between the original figure and the transformed image. Among the isometric transformations, it is considered the most difficult for students to understand (Gomes, 2012; Moyer, 1978), especially when the center of rotation is external to the figure (Gaspar; Cabrita, 2014; Küchemann, 1981); however, its understanding is essential for the development of Geometric Thinking, including intuitive imagination (Jones, 2020), visual perception and spatial reasoning (Gomes, 2012), which helps students interpret the world around them.

When we think about the ways in which students learn, we understand that this learning occurs associated with tasks for students, which can be understood in different ways, according to the different types of tasks (Ponte, 2005) – open, closed, problems, investigations. In relation to problems (and investigations, as broader problems), we can consider a four-step structure for their resolution (Polya, 1975). It is essential that these types of tasks and steps for problem solving are discussed in teacher training so that they can become something natural in teacher practices and, even today, almost 50 years after Polya 's studies, this idea of sustained practice in problem solving can be understood as an innovation, including in view of students' difficulties in solving problems in different mathematical topics (Francioli; Silva, 2021).

Considering that the teacher's mathematical practice is based on the implementation and discussion of mathematical tasks (see, for example, Mason and Johnston-Wilder, 2006; Ribeiro, Mellone and Jakobsen, 2016) and the need for teachers to have the same type of experiences that they are expected to provide to their students, it is essential that teacher training takes place in the same space as practices that are expected to be implemented with students (Ribeiro; Carrillo; Monteiro, 2012) and, therefore, that is supported by the preparation, implementation and discussion of tasks that contribute to developing the specificity of the teacher's knowledge for their professional practice. It is therefore essential that teacher training enables the creation of bridges that reduce the distance between the mathematics that teachers learned and the mathematics that they are expected to teach their students (Zaslavsky; Leikin, 2004). These tasks and associated opportunities need to consider a focus on mathematical processes (Biza *et al*., 2015) with the aim of enabling teachers to transform their mathematical knowledge into pedagogically oriented mathematical practices (Wasserman *et al*., 2022).

In this sense, a set of innovative approaches is essential that consider a focus on issues of mathematical management in the classroom, enabling discussions in a formative context to come closer to an expected authentic teaching and learning scenario, pursuing the objective of ensuring that they are students' mathematical learning is prioritized and not the mathematical content itself (Mitchell; Marin, 2015) or general pedagogical issues without any relation to learning (Ribeiro, 2018). Considering the centrality of tasks in students' mathematical learning, teacher training must also assume this centrality and consider the specificities of the "professional practice" of each of those involved (students and teachers) and the specificities of the teacher's professional knowledge for this practice mathematics to enable students to understand mathematics.

These specificities of teacher practice have been understood from a perspective that assumes the centrality of general pedagogical knowledge – without any reference to the contents covered (see Shulman, 1986, 1987) – and which can be understood as a way of differentiating the “teacher cluster ” of all other “professional clusters”, but staying in these generalities does little or nothing to help us think about the specificities of mathematics teacher practice in relation to other teachers from other areas of knowledge (Ribeiro, 2018). In order to direct attention to these specificities, it is essential to consider what makes the professional practice of mathematics teachers unique. This uniqueness is associated with their professional knowledge to teach mathematics and the fact that this knowledge is considered unique and specific for this professional activity – some of the theoretical perspectives that assume this idea are, for example, *Mathematical Knowledge for Teaching* – MKT (Ball; Thames; Phelps, 2008), *Knowledge Quartet* – KQ (Rowland *et al*., 2009), *Mathematics for Teaching* – MfT (Davis; Simmt, 2006), *Mathematics Teacher's Specialized Knowledge* – MTSK (Carrillo *et al*., 2018) and Interpretive Knowledge – CI (Jakobsen; Ribeiro; Mellone, 2014). In the scope of this work, we assume the *Mathematics conceptualizations Teacher's Specialized Knowledge* and Interpretive Knowledge, which assume that the teacher's knowledge is specialized in the mathematical and pedagogical domain.

MTSK is a conceptualization of the mathematics teacher's knowledge and allows (search) to characterize in detail the specificities of the content of this knowledge considering two domains: *Mathematical Knowledge* (MK) and *Pedagogical Content Knowledge* (PCK). We will discuss here only the content of the MK [4]that is subdivided in three subdomains: *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KSM) and *Knowledge of Practices in Mathematics* (KPM). To bring examples of the content of this knowledge, we chose to focus on rotation, as it is a problematic topic in aspects related to teaching and learning (see, for example, Gaspar and Cabrita, 2014 or Küchemann, 1981).

KoT corresponds to the teacher's mathematical knowledge regarding the mathematical topics to be taught, including procedural and conceptual knowledge, as well as propositions, examples, intraconceptual connections, formulas and algorithms, consequently their demonstrations and the meanings that are associated with the knowledge of phenomenology of each topic (Liñan; Contreras; Barrera, 2016). Four categories of knowledge are considered: (i) procedures; (ii) definitions, properties and foundations; (iii) representation records; (iv) phenomenology and applications.

[4]For more information on the content of PCK in this conceptualization, see, for example, Ribeiro and Almeida (2022) and Ribeiro, Alves and Gibim (2023), which also illustrate an innovative perspective in terms of the form and focus of research dialogue discussion and proposals for teachers.

(i) procedures refer to the set of sequential actions carried out to obtain an answer to a given problem, which may be through algorithms (conventional or alternative) or using other strategies. In the context of rotation, for example, it is related to knowing that, to identify the center of rotation of an already transformed figure, it is necessary to draw the perpendicular bisector between a point on the original figure and its corresponding point in the image, repeating this procedure (at least) twice, in order to obtain the point of intersection of the drawn perpendicular bisectors, which corresponds to the center of rotation.

The (ii) definitions include knowledge about the minimum set of properties of the topic that allow it to be uniquely identified (Liñan; Contreras; Barrera, 2016). It involves knowing that a possible definition of rotation is:

> Let O be a point taken in the plane Π and $\alpha = A\hat{O}B$ a vertex angle O. The angle α *rotation* around the point O is the function $\rho_{O,\alpha}: \Pi \rightarrow \Pi$ thus defined: $\rho_{O,\alpha}(O) = O$ and, for every point $X \neq O$ in Π, $\rho_{O,\alpha}(X) = X'$ is the point in the plane Π such that
>
> $$d(X,O) = d(X',O), X\hat{O}X' = \alpha$$
>
> and the "direction of rotation" from A to B is the same as from X to X' (Lima, 1996, p. 21-22, our translation).

When considering (ii) properties, the associated teacher's knowledge is assumed to know the set of all mathematical attributes that are common to the topic. It included knowing that the composition of two rotations with the same center of rotation is commutative, as well as the composition of two rotations with different centers is not commutative (Breda *et al*., 2011).

The (ii) fundamentals relate to knowledge about the set of mathematical attributes that "support" the topic and connect concepts (Camacho; Guerrero, 2019). Regarding rotation, it refers to knowing that its foundations are the original figure, the center and the angle of rotation (amplitude and direction).

In (iii) representation registers, they include knowing the different ways of representing a topic, concept, process or procedure (Liñan; Contreras; Barrera, 2016), which can be arithmetic, concrete, graphic, pictorial registers, involving verbal or symbolic language (Duval, 1996). It involves knowing that the rotation of a triangle with vertices X, Y and Z with a center of rotation at O from an angle of 60° in a counterclockwise direction can be represented algebraically by $R_o\ [(X, Y, Z), 90°]$.

(iv) phenomenology and applications relate to knowing the concepts associated with a given topic and the different phenomena that involve it, as well as the meaning of each of the possible manifestations and interpretations of these phenomena, according to the different contexts for teaching it (Liñan; Contreras; Barrera, 2016). As an example of knowledge related to the phenomenology of rotation, rotation is an isometric geometric transformation in which a transformation (phenomenon) takes place in the figure.

The KSM subdomain refers to the knowledge of the different connections between mathematical topics (Carrillo *et al*., 2018), considering the temporal aspects of mathematical sequencing: (i) complexification connections and (ii) simplification connections; and the aspects of each topic: (iii) transversal connections and (iv) auxiliary connections (Montes; Climent, 2016).

Complexification connections (i) involve knowledge that enables the teacher to make relationships with other more advanced mathematical topics than is required by the school context. In the scope of rotation, it refers to knowing the complex connection between rotation and the trigonometric circle, since, through the rotation of the right triangle in the trigonometric circle, it is possible to reduce the trigonometric ratios from the 3rd quadrant to the 1st. th quadrant.

(ii) simplification connections refer to the knowledge that allows the teacher to include in the discussion a simpler topic or concept than is required by the school context. It involves knowing the connection between rotation and angle, in which rotating a figure from an angle of 90° is equivalent to a rotation of $\frac{1}{4}$ back in the figure.

With regard to (iii) transversal connections, these relate to knowledge of the nature of some concepts, which emerge when approaching different concepts throughout school mathematics. As an example of a transversal connection between rotation and symmetry, the image obtained after the transformation is symmetric, as symmetry is a concept transversal to isometric geometric transformations.

In relation to (iv) auxiliary connections, they refer to mathematical connections involving different topics, which are not the focus of the discussion, adding an element to contribute and support the mathematical discussion. As an example, the auxiliary connection between rotation and location of points involves knowing that, to perform rotation, it is necessary to identify the center of rotation, which is a point that can be located in the Cartesian plane.

KPM refers to knowledge of the practice of producing mathematics, its functioning and not how to teach it, involving classification and planning, forms of validation and demonstration, the role of symbols, formal language and necessary and sufficient conditions to generate definitions (Carrillo *et al*., 2018). It includes knowledge of the use and functioning of examples and counterexamples (Flores-Medrano, 2016) and how to demonstrate, justify, make deductions and inductions (Carrillo *et al*., 2018). In the context of rotation, it refers to knowing that a counterexample of rotation is axial reflection, as axial reflection is carried out in relation to a straight line called the axis of reflection and not according to an angle. Therefore, the procedures used to perform reflection are different from the procedures used to perform rotation.

This mathematical knowledge underpins the professional practice of the mathematics teacher who seeks to enable students to understand what they do and why they do it at each moment and, from the perspective we assume, this requires considering as a starting point what and how students know of each of the mathematical topics that they have the right and duty to know and understand. The specialized mathematical knowledge for this interpretative practice is called Interpretive Knowledge – CI (Ribeiro; Mellone; Jakobsen, 2013; Di Martino; Mellone; Ribeiro, 2020; Mellone *et al*., 2020).

The importance of assuming as a starting point what and how students know as the premise of IC is essential for an effective ethical discussion in the classroom (Mellone *et al*., 2023), which is one of the challenges in the field of Mathematics Education (Radford, 2021), and involves ensuring that mathematical discussions are associated with opportunities for inclusion, commitment and respect, from the same perspective of community ethics (Radford, 2021), for a teaching approach aimed at understanding mathematics.

According to the Springer Nature Encyclopedia, Interpretive Knowledge:

> It refers to broad and deep mathematical knowledge that allows teachers to support students in developing their own mathematical knowledge, taking their own reasoning and productions as a starting point, regardless of whether they are non-standard or incorrect. CI complements students' knowledge of typical errors or strategies, with knowledge of possible origins of typical and atypical errors and knowledge of the use of errors as an effective source of learning (Di Martino; Mellone; Ribeiro, 2020, p. 426, our translation).

CI allows the teacher to understand the mathematics that supports the students' reasoning and ways of thinking present in their productions, in order to explore errors, understood as learning opportunities (Borasi, 1987) and provide guidance based on the meaning attributed. In this Knowledge that supports interpretative mathematical practice, two central notions are considered: solution space and *feedback*.

The solution space refers to the set of multiple forms and representations that each individual conceives when asked to solve a problem – even if this problem has a single solution (Jakobsen; Ribeiro; Mellone, 2014). It is essential that the teacher knows different ways of proceeding to solve a problem so that, when faced with a student's production that is different from his own, he does not have difficulties in interpreting it and does not consider it as incorrect just because it is different from his own - it is necessary, we therefore, have a solution space with a multiplicity of elements.

After understanding and interpreting the production, the teacher must propose guidance to the student, which is configured as *feedback* – a form of communication and interaction between teacher and student (see, for example, Black and William, 1998 or Hattie and Timperley, 2007). There are different types of *feedback* and, when the teacher aims to explore the mathematical reasoning present in the production (Santos; Pinto, 2009), proposing clear guidelines that encourage the student to review their production, rethink the strategies used and develop their mathematical understanding, it is of constructive *feedback* (Di Martino *et al*., 2017). Other types of *feedback* (Galleguillos; Ribeiro, 2019) are: (i) *feedback* on how to solve the problem – instructive guidance on procedures to be followed to solve a specific problem; (ii) confusing *feedback* – although correct, it is incomprehensible to the student due to the complexity of the instructions; (iii) counterexample as *feedback* – contains an explanatory example of why the student's solution is incorrect; (iv) superficial *feedback* – insufficient or inconsistent guidance, which does not help the student understand their mistakes.

Categories (i) and (ii) are associated with an instructive practice, explaining to the student how to proceed, which does not require the teacher to attribute meaning to the students' mathematical thinking, imposing their way of doing things. Categories (iii) and (iv) are associated with evaluative practices and focus on explaining why students' production contains errors, but they demand from the teacher a correct interpretation of the production, requiring mathematical knowledge that allows the teacher to approach a problem in different ways and involves him knowing several examples so that he can explain why some ways of proceeding are incorrect.

In the context of rotation, considering a task in which it is requested to identify some point that remains fixed, when the movement (rotation) is carried out and a student's production that expresses "there are no fixed points" (Silva; Ribeiro, 2023), an example of evaluative *feedback* involves the teacher simply evaluating the production as incorrect, as he did not identify the fixed point that is the center of rotation and indicating the correct point in the production. Constructive feedback must consider identifying the center of rotation, proposing, for example, to the student, guidance to draw some perpendicular bisectors between some points of the figure and their corresponding points in the image, with the aim of having him review his production and identify the *center* of rotation, with this orientation associated with questions about what happens to all the perpendicular bisectors and whether they intersect, thus enabling the student to realize that they intersect at a single common point, which is the center of rotation.

This *feedback* is associated with, and is conditioned by, the level of knowledge that the teacher holds and on which, in CI, three levels are defined (Mellone *et al*., 2017): (i) evaluative interpretation; (ii) interpretation for teaching practice; (iii) interpretation as research.

(i) evaluative interpretation is associated with the lowest level of IC that leads the teacher to establish a correspondence between his production and that of the student, considering only his way of proceeding as correct and any production that differs from his is evaluated as incorrect. (ii) interpretation for teaching practice is based on an intermediate level of IC and corresponds to the teacher considering what is expressed in the student's production, to plan the next discussions to be proposed and achieve the mathematical learning objectives; therefore, it takes as its starting point what and how students reveal they know. Considering a higher level of IC, we have (iii) interpretation as research that refers to the teacher reviewing his own mathematical formalization, making the student's production a source of research, even if these productions seem different from what is traditionally taught in schools, since, in this interpretative practice, the teacher can discuss the student's production with colleagues and even research other ways of proceeding, which makes it possible to learn about other ways of doing mathematics and solving a given problem, resulting in the expansion of your solution space.

To propose constructive *feedback*, a high level of IC is required from the teacher, and the development of the teacher's specialized knowledge demands training contexts (Ribeiro; Mellone; Jakobsen, 2013) in which Training Tasks are implemented and discussed (Ribeiro; Almeida; Mellone, 2021).

There are different perspectives on Tasks for teachers, such as Professional Learning Tasks (Smith, 2001; Ribeiro; Ponte, 2020) or formative tasks (Martín *et al*., 2023). Since our focus is on the development of the teacher's knowledge and not on their learning, the tasks are understood as a specializing resource for professional practice, therefore called Tasks for Training - TpF (Ribeiro; Almeida; Mellone, 2021) are specific to the development of this specialized knowledge of the teacher.

The TpF form part of a set of documents that are prepared to support the training to be carried out and which corresponds, in the conceptualization developed in the CIEspMat group[5], to the so-called Formative Task (Ribeiro; Almeida; Mellone, 2021) which is composed of four documents: (i) Training Task; (ii) document with the five central dimensions for implementing the task in the classroom; (iii) teacher's document and (iv) trainer's document.

(i) Training Task: task to be delivered to teachers in training contexts and conceptualized to access and develop the Interpretive and Specialized Knowledge of trainees. For its conceptualization, the most recent research results and results of national and international tests are considered, which identify the most problematic mathematical topics for students (and, therefore, also for teachers) – in which, for example, the Problem solving and formulation are not topics, but considered contexts of and for discussion of mathematical topics, and is structured in two or three parts. All parties are associated with the objectives of accessing and developing the teacher's knowledge, and this access is related to the specialized pedagogical approach to implementation and to the research that always occurs in training contexts, considering research and training in an intertwined way. The Preliminary part focuses on some dimension of mathematical or pedagogical knowledge and seeks to establish a starting point for the discussions to be carried out – what and how the teacher knows about the topic, what he already does in his mathematical practice and how he does it. Part I is structured around a task for the student which the teacher is expected to implement in their practice, but it also includes a set of questions emerging from the problems identified in the literature on teacher knowledge and which are formulated in line with to the content of certain subdomain(s) of MTSK in order to focus discussions.

It is important to note that this is a training option that allows you to primarily direct the focus of attention to the specificities of mathematical practice and the specialized knowledge that supports this practice, and that, despite this directional focus, the implementation of TpF allows, through the experience, teachers can carry out discussions involving all subdomains of their specialized knowledge.

When the TpF contains a part II, its objective is to develop Interpretive Knowledge and is called Interpretive Task – TI (Mellone *et al*., 2020). In this part II, some contexts of student or teacher productions are included (written, video, classroom discussions, discussions in

[5]CIEspMat is a Research and Training group that develops work focused on developing the Interpretive and Specialized Knowledge of teachers and future teachers of and who teaches mathematics – from Early Childhood Education to High School. Available at: www.ciespmat.com.br. Accessed on: 10 Dec. 2023.

training contexts) chosen because they are mathematically powerful for developing IC and for attributing meaning to the forms of thinking that support these productions and proposing constructive *feedback*.

(ii) document with the five dimensions: set of central indications for the teacher to implement, discuss and achieve the mathematical learning objectives of the task for the student (for example Ribeiro and Torrezan, 2022 or Silva and Ribeiro, 2023): (1) Mathematical learning objective pursued with the task; (2) Required resources and way(s) of student work; (3) Skill from the National Common Curricular Base (Brazil, 2018) associated with the task; (4) Possible difficulties of students; (5) Comments for implementation and associated mathematical discussions.

(iii) teacher's document: encompasses all the central elements of the specialized mathematical knowledge of the topic, considering the conceptualization of MTSK, addressed in TpF, which aims to develop in teachers participating in the training.

(iv) trainer's document: contains a set of guidelines so that the trainer can implement TpF, minimizing deviations from the training objectives associated with its conceptualization, considering the associated research intention. It therefore contains the training and research objectives, as well as a set of indications relating to the specificities of the training that is intended to be carried out, detailing the objectives of each TpF question and the Specialized and Interpretive Knowledge that is expected to be developed, as well as pedagogical indications implementation specifics that are associated with possibilities of replicability in practice contexts with students – or games with children in Early Childhood Education. It also includes examples of questions and possible answers of the knowledge involved and required and the discussions to be held at each stage[6].

This central triad for innovation that we consider in research, practice and training is composed of these two previous blocks of Knowledge (ways of understanding the teacher's knowledge and its specificities for practice, training and research) and resources for practice, training and collection of information for research, is only complete with a pedagogical implementation and methodological approach that maximizes and enhances the quality of discussions, the sustainability of the development of the teacher's specialized knowledge and associated research.

This specialized pedagogical and methodological approach that we have developed and which impacts the foci of discussion that are considered, assumes two types of replicable

[6]For some examples, see Ribeiro, Alves and Gibim (2023) or Ribeiro and Torrezan (2022).

structure: Individual-Collective-Individual Cycle – ICI (Pacelli *et al*., 2020) or Small Group-Collective- Small group – Pg - C-Pg (Jakobsen; Ribeiro; Mellone, 2022; Mellone *et al*., 2023). The difference between these structures is in the form of TpF resolution work, as in the ICI Cycle teachers resolve part of the TpF individually, followed by a collective discussion in a large group that seeks to synthesize the ways of Thinking that emerged individually and in which everyone become responsible for the knowledge developed in this context and, after about a month, participants must send their "revised and improved" answers to the same TpF so that some elements still necessary for further development can be identified and a knowledge analysis carried out developed.

An adaptation to this approach considers the fact that individual resolution of the TpF does not necessarily enhance the development of broad and deep Interpretive Knowledge (Jakobsen; Ribeiro; Mellone, 2022). Thus, in the Pg - C-Pg Cycle, teachers are organized into groups (ideally four participants) to discuss, reflect and resolve the TpF and the two subsequent moments follow the same previous structure. This option is also associated with the need to enable teachers to experience group work in the first person so that they can carry out the same type of discussion with their students in their practices.

An example of a Training (Interpretive) Task associated with innovations

Training Tasks can have different structures and here we focus our attention on Interpretive Tasks (IT) that seek to more specifically develop the Interpretive Knowledge of (future) teachers. This example intends to illustrate the conceptualization of the training resource and instrument for collecting research information, and for this we present an IT within the scope of the rotation and, subsequently, we carry out a discussion of the reasons that lead to the inclusion of students' questions and productions – considering the three dimensions of innovation: theoretical, resources, implementation.

Figure 1 – Interpretive Task within the scope of the rotation

Part Preliminary

1. Imagine you are on the street and someone asks you: In a mathematical context, what is rotation? What would you answer? (Do not forget that we are on the street and therefore do not intend to teach this person).

2. Professor Mário intends to discuss the mathematical definition of rotation with his 7th grade students. He found some definitions and will take them to discuss in a training session under the responsibility of CIEspMat, as he needs help to know which definition(s) is/are and which one (s) will be most appropriate to discuss with his/her colleagues.
   Help Professor Mário choose the most appropriate definition(s) presented below and justify why they are definitions, or not, indicating what changes would have to be made to make them so.

Rotation definitions found by Professor Mário:

(A) In a rotation, every figure is rotated with respect to a point called the center of rotation. The original and rotated figures have the same measurements, and the elements of the original and rotated figures are the same distance from the center of rotation.

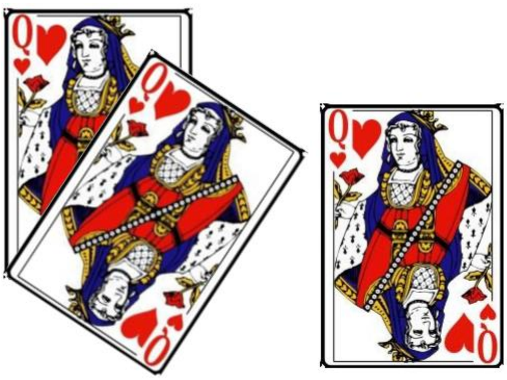
(B) Rotational symmetry occurs when a flat figure is rotated around a point, according to an angle (with an opening measurement between 0º and 360º), in a certain direction (clockwise or counterclockwise). With this, we always obtain a flat figure that maintains the same shape and size as the original figure.

Part I

Task: Rotated letters[7]

(You should always explain your reasoning by describing the process you use to answer the question. You can do this using diagrams, words, calculations, ...)

Observe the Situations with cards from the "queen" deck:



Situação 2

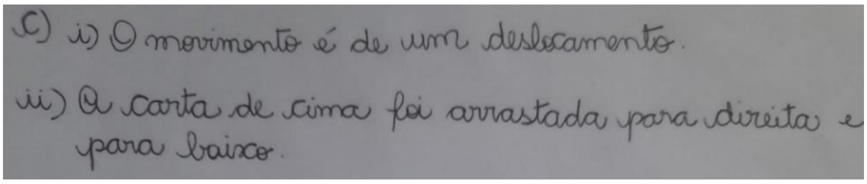
[7]Adapted from Paques and Oliveira (2012).

a) Record what caught your attention when looking at the cards for each of the Situations.
b) In Situation 1, can you identify the movement made to build the entire card from one of its parts? Justify.
c) In Situation 2:
   i) Can you identify the movement made to obtain the new license? If yes, describe it. If not, justify.
   ii) Explain the procedures that can be carried out to obtain the new image.
d) In each situation, can you identify a point that remains fixed when the movement is made? Justify.

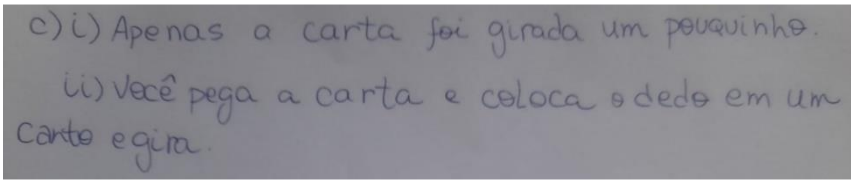
1. Consider the previous task:
   (i) Solve the task by yourself, without thinking about a teaching context.
   (ii) What do you think will be the students' greatest mathematical difficulties in solving this task? Justify your response.
   (iii) What do students already need to know to carry out this task? Justify your response.

Part II

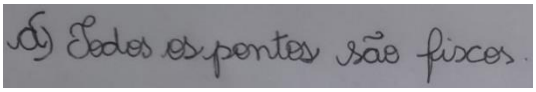
1. After implementing this task with his 7th year D students, Professor Mário obtained some answers and decided to also take them to discuss during the training of CIEspMat. See the productions of students Aline and Camila regarding questions c) and d) of the student task:


c) i) O movimento é de um deslocamento.
ii) A carta de cima foi arrastada para direita e para baixo.

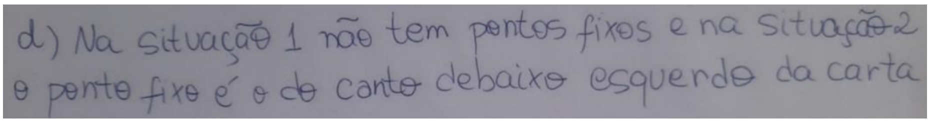
Aline's production for question c).


c) i) Apenas a carta foi girada um pouquinho.
ii) Você pega a carta e coloca o dedo em um canto e gira.

Camila's production for question c).


d) Todos os pontos são fixos.

Aline's production for question d).


d) Na situação 1 não tem pontos fixos e na situação 2 o ponto fixo é o do canto debaixo esquerdo da carta

Camila's production for question d).

a) For each of the productions, indicate whether you consider them mathematically correct (adequate) or not, justifying the mathematical reasoning evidenced.
b) For each of the students, provide constructive *feedback* (rather than saying whether it is correct or incorrect, the teacher must attribute meaning to the students' resolutions in order to later help in the development of their mathematical knowledge).

Source: Prepared by the authors

In the preliminary part, two questions are included here aimed at accessing and developing the content of the teacher's KoT. In question 1, the aim is to access (and develop through subsequent discussions) the teacher's knowledge associated with the phenomenology of the rotation topic - placing the question in a "typical non-educational" context aims to remove the teacher from a "typical" context explain how you would do it in the classroom", as the aim is to access your specialized mathematical knowledge and not your pedagogical approaches.

In question 2, the focus is on the teacher's knowledge associated with what he assumes to be a mathematical definition (Zazkis; Leikin, 2008) – mathematically valid – and that is understandable to his students. It also seeks to promote critical reflection on the " pseudo definitions" found in many pedagogical materials (here textbooks) and on the need for knowledge that allows these pedagogical proposals to be improved for discussion in the classroom, also through discussion this question, knowing that there are different mathematical definitions for the same mathematical entity. This inclusion considers the need to take into account that students have difficulties in interpreting and using definitions (Mariotti; Fischbein, 1997; Zazkis; Leikin, 2008), and it is essential that the teacher chooses didactically appropriate definitions for the students' age group and the teaching context, taking as a starting point definitions that consider what students already know.

In part I, a task for 7th year students (12 or 13 years old) is included within a rectangle – in accordance with official Brazilian curriculum documents (Brazil, 2018) and three questions for teachers. This task for students pursues the objective of mathematical learning (part of the five dimensions): developing students' understanding of the isometric geometric transformation rotation, with regard to identifying its constituent elements and procedures carried out to carry out the rotation, from rotated images.

It is worth noting that tasks for students are always formulated considering the greatest difficulties identified in research results. Although, in the context of rotation, these difficulties are associated, for example, with identifying the center of rotation, especially when it does not belong to the figure (Gaspar; Cabrita, 2014; Küchemann, 1981), in this task, as it is an introduction (Ribeiro; Almeida; Mellone, 2021) and taking as a starting point something that teachers know – a "typical teaching material" task – the option was taken to include examples whose centers of rotation belong to the figure, as the objective is not is to make the task difficult for the student, but to develop their mathematical understanding and their ways of thinking mathematically.

In the questions for the teacher, by asking them to solve the task themselves (question (i)) the aim is to access mathematical knowledge at the students' level of knowledge (solve the same task that the students are expected to solve). In this specific case, it is associated with correctly identifying the movement carried out a)); the procedures carried out to obtain the image through rotation b) and c) – (i)); differentiate rotation from other isometric transformations c) – (ii)); procedures associated with rotation and the constituent elements that determine this transformation (d).

By asking teachers to identify the students' greatest mathematical difficulties in solving this task (question (ii)), the aim is to start the movement of getting teachers to establish the mental habit of anticipating their students' possible answers, considering them for the planning and implementation of mathematical discussions. This anticipation is also associated with the focus intended in part II in order to help identify and attribute meanings to students' errors and their ways of thinking mathematically. With question (iii), the aim is to access and develop the teacher's knowledge regarding what students know (what and how they know or should know) that would support the completion of the task (question 1 – (iii)). This includes, for example, knowing the notion of angle, associated with the amplitude and direction of the angle of rotation and, from this, discussing with students the procedures to be carried out to measure the amplitude of an angle, which Possibly, it would include revisiting and questioning students about what they know about the use of the protractor, always from a questioning perspective and not “giving the rule”. Furthermore, if students already know the central reflection, the teacher can problematize the equivalence between central reflection and the 180º rotation (Bairral; Silva, 2010) considering Situation 1 of the task for the student.

In part II, the focus is on the teacher's Interpretive Knowledge. To this end, in this task, several student productions are included for the student's task in part I and the teacher is asked to interpret and attribute meaning to the ways of thinking and proceeding in mathematics that support these productions, providing constructive *feedback to* each student. The questions seek to access the level of Interpretive Knowledge and, through subsequent discussions, promote a change in the level of this knowledge. Associated with the implementation of IT implementing the Pg - C-Pg Cycle, we have a brief document that discusses what is and is not constructive *feedback*, as the type and nature of this *feedback* is associated with the CI levels revealed by teachers. Let us note that the way we conceive the role and knowledge of the teacher (Almeida; Ribeiro; Fiorentini, 2021) is related to the understanding of how the trainer himself plans and implements his training practices (Ferreira; Behrens; Teixeira, 2019), which can be generalist or aimed at developing the specificities of the teacher's knowledge.

In this part, the productions of the students included are of fundamental importance and their selection (or elaboration based on research results) is associated with the specificities of the training intentionality that is considered. Each of them is included because it is associated with a specific mathematical discussion and, simultaneously, together, these productions need to enable a change in the level of knowledge that demands developing an understanding of the phenomenon of rotation. These students' productions, here focusing on errors, are associated with a change in conceptions regarding errors (Borasi, 1987) and their pedagogical use as a starting point for the development of students' knowledge and the context is associated with the development of habit of developing an interpretative mathematical practice based on attributing meaning to the mathematical reasons that support students' productions, whether they are inadequate or contain unexpected approaches,[8] so that the teacher rethinks their own mathematical formalization and expands their solution space (Ribeiro, 2024) , – can incorporate a greater number of elements into this solution space.

Aline's production for question c) was included, as it presents an incomplete answer to the movement carried out, expressing the rotation only as a displacement (in (i)), without specifying that this displacement is in relation to an angle, with the The term displacement can also be used to refer to translation; considers movement as two translations (question (ii)), making it possible to discuss the difference between isometric geometric transformations – in addition to the names –, such as the procedures (algorithms) involved and the result obtained (image). In d) it does not identify that a movement was carried out to obtain the entire letter from one of its halves, which makes it possible to bring into the discussion the difficulty related to visualizing the rotation already carried out and the lack of understanding that isometric geometric transformations are associated with the idea of a rigid movement that maintains distances and range of angles, implying that the original figure and the image through transformation are congruent.

Camila's production for question c) was included, as it is associated with an understanding of rotation as a turn, but does not specify the amplitude or direction of the rotation angle, these two elements being fundamental for understanding rotation. It also enables a discussion associated with the procedures for performing the rotation and the possibility of its

[8]For example, in Jakobsen , Ribeiro and Mellone (2014) some unexpected productions (which are not part of teachers' usual solution space) are presented and discussed within the scope of rationals.

generalization – therefore configuring the existence of an algorithm. In question d), the production makes it possible to discuss the difficulty and problem in identifying the center of rotation as the only point that remains fixed when performing the rotation – in both situations it belongs to the figure –, but a discussion associated with how to identify the center of rotation by determining the perpendicular bisectors between the points of the original figure and their corresponding points in the image.

By asking teachers to provide constructive *feedback* (question (b)), the aim is to place the teacher in the context of an interpretive practice, encouraging him to propose constructive *feedback* (Di Martino *et al* ., 2017; Mellone *et al*., 2020), going beyond a merely evaluative perspective (see, for example, Ribeiro, 2024). This requires the teacher to effectively "listen" to the students' mathematical thinking, which goes far beyond a direct reading and description of what was recorded (copy) or "sensory listening", and requires listening that, in fact, consider as a starting point what and how students reveal they know and, based on this active listening, propose clear and objective guidelines that help students develop their mathematical understanding.

Some final comments

To innovate, it is necessary to think and do something different from what has been done until now, and this doing differently in innovative ways indicates other possibilities and paths that had not been considered until then, but which are possible and impactful for the associated contexts and objectives. In our context, these innovative forms and approaches already reveal results in previous specific research that seeks to identify what happens at a given moment – taking photos of what happens at each moment (see, for example, Couto and Ribeiro, 2019; Ribeiro, Jakobsen and Mellone 2022) –, which indicate a set of possibilities to "look at each frame" and understand what leads to knowledge being developed and enabling these reasons and approaches to be generalized to other themes and topics.

The ways of understanding the teacher's knowledge specifically related to their professional practice and enabling students to understand mathematics and develop their ways of thinking mathematically (included here in (i) theoretical innovations) is something that breaks with a set of teaching practices research and training that prioritize the teacher's knowledge in general terms (Shulman, 1987; Ribeiro, 2018) and focus training on issues of general pedagogical knowledge without the necessary discussion of mathematical knowledge specifically related to the teacher's professional practice (Fiorentini; Creci, 2017) which will make it possible to change the focus and objectives of this practice to medium and long-term objectives.

On the other hand, resources have been a focus of attention in several researches (and training) in Mathematics Education (Grando, 2015), but there too the focus has been, very often on the resource itself and not on the mathematical discussions that each one of these resources enhances or hinders its impacts on students' discussions and mathematical learning. When considering the Training Tasks themselves, which are conceptualized based on the students' greatest mathematical difficulties and focusing on the specificities of the teacher's knowledge as a resource for their own training and research, we aim for the results to be directed towards mathematical learning and development of the teacher's specialized knowledge. The TpF that was presented illustrated this perspective of (ii) innovation of resources for training and information collection. Innovation of resources for training because, despite considering it a task for students, the objective of training is not "how to implement with students in the classroom", but the discussion prioritizes developing specialized mathematical knowledge that will enable mathematical discussions of a higher level than those that would occur if this mathematical knowledge were limited to "knowing how". The multiplicity of forms and possibilities of how to implement the task with students (specialized pedagogical knowledge) is something that is approached in a transversal way and assuming a perspective that this specialized pedagogical knowledge "is not taught, it is lived", just as it is not Thinking is taught, but ways of developing this thinking are promoted.

Associated with the discussions of experiencing pedagogical knowledge are the approaches to collecting information that seek to contribute to, in an intertwined way, developing specialized knowledge in a sustained way – associated with the third type of innovation (iii) of the methodological approaches of implementation of Training Tasks and conceptualization of Training Tasks – corresponding to the ICI and Pg -G-Pg methodological approaches. Here, due to the stage in which the research associated with Isometric Geometric Transformations and symmetry is at [9](example of the TpF presented), we do not bring examples of the impact of these methodological approaches on the wealth of information collected to access and discuss the specificities of Interpretive and Specialized Knowledge, but we leave

[9]This research has now entered the stage of collecting information in a training context designed associated with these three types of innovation, so we will soon have results on the applicability and impact of the three dimensions for research and training.

some open questions that could be the focus of research that helps us advance the knowledge we have on the topic and the teacher's mathematical knowledge and practice.

Thus, some emerging questions that can open a research agenda with this specialized focus on knowledge, tasks and methodological approaches are:

(i) What Interpretive Knowledge do teachers reveal when interpreting and attributing meaning to students' productions?

(ii) What levels of Interpretive Knowledge can we identify throughout training and how do these levels change throughout the year in relation to the Training Tasks and discussions developed?

(iii) What are the characteristics of the Training Tasks that maximize the development of the specificities of the teacher's knowledge?

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