RPGE Revista on line de Política e Gestão Educacional

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# FOSTERING UNDERSTANDING OF INFINITE GEOMETRIC SERIES THROUGH A VISUAL AREA MODEL: A DIDACTIC DESIGN BASED ON APOS THEORY

## *PROMOVENDO A COMPREENSÃO DAS SÉRIES GEOMÉTRICAS INFINITAS POR MEIO DE UM MODELO VISUAL DE ÁREAS: UM PROJETO DIDÁTICO BASEADO NA TEORIA APOS*

## *FOMENTAR LA COMPRENSIÓN DE LAS SERIES GEOMÉTRICAS INFINITAS MEDIANTE UN MODELO VISUAL DE ÁREAS: UN DISEÑO DIDÁCTICO BASADO EN LA TEORÍA APOS*

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**How to reference this paper**:

Nam, P. S., Tu, P. T. T., Hau, N. H., & Nga, T. T. (2026). Fostering understanding of infinite geometric series through a visual area model: a didactic design based on APOS theory. *Revista on line de Política e Gestão Educacional*, *30*(esp2), e026074. https://doi.org/10.22633/rpge.v30iesp2.21364

| **Submitted**: 26/04/2026
| **Revisions required**: 02/05/2026
| **Approved**: 12/06/2026
| **Published**: 01/08/2026

**Editor**: Prof. Dr. Sebastião de Souza Lemes
**Deputy Executive Editor**: Prof. Dr. José Anderson Santos Cruz



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**ABSTRACT**: This study investigates how a visual area model (iterative unit square subdivision) influences students' construction of the sum of an infinite geometric series, and clarifies the cognitive mechanisms involved. Following a didactic engineering approach, 52 Grade 11 students completed a four-question worksheet. Data from worksheets and semi-structured interviews were analyzed using APOS-based coding. The visual model appeared to facilitate completion (36.5% of students correctly predicted the limit 1/3 before formal instruction) and encapsulation (61.5% reached the Object level). However, generalization to arbitrary ratios remained challenging (only 34.6% achieved Schema). The study extends APOS theory by demonstrating that visual totality alone can trigger completion and encapsulation, even without full algebraic mastery. It also offers a refinement of the genetic decomposition for infinite geometric series, highlighting the need for additional scaffolding in generalization. A ready-to-use instructional design is provided.

**KEYWORDS**: Infinite geometric series. APOS theory. Area model. Didactic engineering. Visual representation.

***RESUMO***: *Este estudo investiga como um modelo visual de área (subdivisão iterativa do quadrado unitário) influencia a construção, pelos alunos, da soma de uma série geométrica infinita, e esclarece os mecanismos cognitivos envolvidos. Seguindo uma abordagem de engenharia didática, 52 alunos do 11º ano preencheram uma ficha de exercícios com quatro questões. Os dados das fichas de exercícios e das entrevistas semiestruturadas foram analisados utilizando codificação baseada no APOS. O modelo visual pareceu facilitar a conclusão (36,5% dos alunos previram corretamente o limite 1/3 antes da instrução formal) e o encapsulamento (61,5% alcançaram o nível Objeto). No entanto, a generalização para razões arbitrárias continuou sendo um desafio (apenas 34,6% alcançaram o nível Esquema). O estudo amplia a teoria APOS ao demonstrar que a totalidade visual por si só pode desencadear a conclusão e o encapsulamento, mesmo sem domínio algébrico completo. Ele também oferece um refinamento da decomposição genética para séries geométricas infinitas, destacando a necessidade de suporte adicional na generalização. É fornecido um projeto instrucional pronto para uso.*

***PALAVRAS-CHAVE***: *Séries geométricas infinitas. Teoria APOS. Modelo de área. Engenharia didática. Representação visual.*

***RESUMEN***: *Este estudio investiga cómo un modelo visual de área (subdivisión iterativa del cuadrado unitario) influye en la construcción que hacen los estudiantes de la suma de una serie geométrica infinita, y aclara los mecanismos cognitivos implicados. Siguiendo un enfoque de ingeniería didáctica, 52 estudiantes de 11.º grado completaron una hoja de ejercicios de cuatro preguntas. Los datos de las hojas de trabajo y las entrevistas semiestructuradas se analizaron utilizando una codificación basada en APOS. El modelo visual pareció facilitar la finalización (el 36,5 % de los estudiantes predijo correctamente el límite 1/3 antes de la instrucción formal) y la encapsulación (el 61,5 % alcanzó el nivel Objeto). Sin embargo, la generalización a razones arbitrarias siguió siendo un desafío (solo el 34,6 % alcanzó el nivel Esquema). El estudio amplía la teoría APOS al demostrar que la totalidad visual por sí sola puede desencadenar la finalización y la encapsulación, incluso sin un dominio algebraico completo. También ofrece un refinamiento de la descomposición genética para series geométricas infinitas, destacando la necesidad de andamiaje adicional en la generalización. Se proporciona un diseño instruccional listo para usar.*

***PALABRAS CLAVE***: *Series geométricas infinitas. Teoría APOS. Modelo de área. Ingeniería didáctica. Representación visual.*

## INTRODUCTION

The concept of the sum of an infinite geometric series is highly abstract and poses a challenging topic for learners, largely due to the conflict between different foundations of mathematical thinking (Nam et al., 2024). Traditional algebraic thinking typically revolves around objects that are “finite”, “discrete”, and “static,” whereas knowledge of limits and infinite sums requires analytical thinking characterized by “infinity,” “continuity,” and “variation” (Nam, 2013). This “infinite” and “dynamic” nature constitute a core barrier: the sum of an infinite geometric series becomes extremely difficult to grasp if approached only through conventional static methods (Nam et al., 2024). When students encounter higher-level mathematical ideas such as the sum of an infinite geometric series (involving limits, infinity, and closed-form formulas), they frequently struggle to transform their cognitive structures. They tend to apply algorithms mechanically rather than developing a deep conceptual understanding (Arnon et al., 2014). A key obstacle lies in the intuitive notion of potential infinity— viewing infinity as an ongoing, never-ending process. This intuition hinders students from accepting actual infinity—seeing an infinite process as a completed, static totality (Fischbein, 2001; Villabona et al., 2024).

To address these cognitive barriers, this study proposes using a visual area model—specifically, an iterative unit square subdivision and coloring task—as an introductory activity. Such infinite iterative geometric models (e.g., Sierpiński triangle, nested squares) provide a concrete context for students to perform actions on infinite processes, compare finite stages with the limiting state, and explore different facets of the concept of infinity (Villabona et al., 2024).

Despite the potential of visual models, few studies have systematically investigated how an iterative area model can support the transition from Action to Object for the specific case of the sum of an infinite geometric series. In particular, the mechanisms of completion and encapsulation have been discussed theoretically (Villabona et al., 2024) but rarely examined empirically in a high school setting with a bounded visual representation. Furthermore, it remains unclear whether a visual totality alone can trigger encapsulation without the need for formal algebraic limit manipulations. Several studies have used dynamic geometry software or PowerPoint to create visualizations (e.g., Nam et al., 2024), but they have not analyzed the underlying cognitive processes through a framework such as APOS. This study addresses these gaps by applying APOS theory within a didactic engineering framework to a unit square area model.

This study aims to design and evaluate a teaching activity based on APOS theory (Action–Process–Object–Schema) to help students construct a robust understanding of the sum of an infinite geometric series. By applying a genetic decomposition from APOS, we design pedagogical strategies that support students in interiorizing processes and encapsulating them into objects, thereby moving from viewing an infinite geometric series as an "Action" or "Process" towards a "Totality" and finally a static "Object". The study addresses two research questions:

RQ1: According to APOS theory, what mental structures (Action, Process, Object, Schema) and cognitive mechanisms (interiorization, encapsulation, completion) enable students to successfully operate on infinite entities when learning the sum of an infinite geometric series?

RQ2: How does the use of a concrete visual area model (iterative unit square subdivision) affect students' encapsulation of an infinite process and their completion mechanism, helping them achieve a Totality and Object conception?

This study makes three distinct contributions:

Theoretical contribution: It extends the understanding of completion and encapsulation mechanisms in the context of infinite geometric series, providing empirical evidence that a bounded visual totality can trigger completion even before formal limit instruction, and that encapsulation can occur via visual reasoning alone for some students.

Practical contribution: It offers a ready-to-use, low-cost instructional design (a four-question worksheet based on a unit square) that connects visual, algebraic, and limit representations, which can be easily implemented in high school classrooms.

Methodological contribution: It demonstrates the integration of didactic engineering with APOS theory, showing how a priori analysis (genetic decomposition) can guide the design of tasks and how a posteriori comparison with observed outcomes can refine the theoretical framework.

## THEORETICAL FRAMEWORK

### *APOS theory: key constructs and mechanisms*

APOS theory is a constructivist framework that seeks to explain how learners build mathematical knowledge through stages of mental structures: Action, Process, Object, and Schema (Arnon et al., 2014). Two key cognitive mechanisms drive transitions:

Interiorization: transforming an Action (externally guided step-by-step manipulation) into a Process (mental repetition, ability to reverse or combine steps, no need for external stimuli).

Encapsulation: turning a Process into a static Object (a totality that can be acted upon). For infinite processes, this often yields a Transcendent Object—a completed infinite entity that may not inherit all properties of its finite approximations (Brown et al., 2010).

Completion: a specific mechanism for infinite iterative processes, where the learner envisions the infinite continuation as a finished whole (Villabona et al., 2024).

These mechanisms are often considered central to understanding how students may move from finite, step-by-step computations to the closed-form sum of an infinite geometric series.

While APOS theory has been applied to various mathematical concepts, some of the specific mechanisms of completion and encapsulation for infinite geometric series remain underexplored, particularly in high school contexts. Most existing APOS studies focus on finite processes or on the limit concept in calculus, but the transition from potential to actual infinity in the specific case of geometric series has not been systematically analyzed through the lens of a visual area model.

### *Difficulties in learning the infinite sum of a geometric sequence*

The concept of the sum of an infinite geometric series is highly abstract and poses a challenging topic for learners, largely due to the conflict between different foundations of mathematical thinking (Nam et al., 2024). Traditional algebraic thinking typically revolves around objects that are “finite,” “discrete,” and “static,” whereas knowledge of limits and infinite sums requires analytical thinking characterized by “infinity,” “continuity,” and “variation” (Nam, 2013). This “infinite” and “dynamic” nature constitute a core barrier: the sum of an infinite geometric series becomes extremely difficult to grasp if approached only through conventional static methods (Nam et al., 2024).

In terms of cognitive structure, the greatest difficulty for students often stems from intuitions related to potential infinity (Fischbein, 2001). Learners tend to visualize infinity as a continuous, ongoing process of motion with no end point, and this very intuition becomes an obstacle that prevents them from accepting the concept of actual infinity—that is, viewing an infinite process as a static entity or a completed totality (Fischbein, 2001; Villabona et al., 2024). When stuck in the Process stage and unable to perform the mechanism of encapsulation to transform that infinite process into a static Object, students fail to perceive the mathematical equilibrium at the limit state (Artigue, 1998; Dubinsky et al., 2013). A typical consequence of this cognitive barrier is that students often regard the limit of an infinite series (such as the sum of a geometric series) merely as an “approximation” that gets closer and closer, rather than accepting it as an exact value (Artigue, 1998; Sierpinska, 1987).

From a different theoretical perspective, Sfard (1991) and Gray & Tall (1994) explained the transition from mathematical processes to concepts through reification and procept theories. Nam et al. (2024) used these frameworks to interpret their PowerPoint-based teaching, arguing that the dynamic triangle model helped students move from performing the iterative process (adding smaller triangles) to recognizing the total sum as a fixed entity ($\frac{1}{3}$). However, reification theory does not explicitly account for the specific mechanisms of completion and encapsulation for infinite processes, nor does it distinguish between potential and actual infinity. APOS theory, with its constructs of completion and encapsulation of infinite processes (Villabona et al., 2024), offers a more fine-grained analysis. The present study therefore adopts APOS to complement and extend the insights from reification-based studies.

Furthermore, the assimilation of this concept is also hindered by the complex nature of Transcendent Objects (Brown et al., 2010; Villabona et al., 2024). Through the lens of APOS theory, the objects obtained from encapsulating an infinite iterative process do not necessarily inherit all the properties of the preceding finite process states (Brown et al., 2010). Students often face difficulties and cognitive conflicts because they attempt to project the attributes of intermediate finite stages (e.g., partial sums always being less than the limit) onto the final infinite limit state, leading them to reject the exact equality (Ely, 2011). This disconnect makes it difficult for learners to logically comprehend why an infinite summation of positive terms (which never stops) can ultimately converge to a completely finite sum.

Beyond these fundamental cognitive obstacles, several specific difficulties have been observed in classroom practice:

- Misunderstanding the limit concept: Students confuse the limit of partial sums with a partial sum itself, believing that the infinite sum “approaches” but never “reaches” the limit.
- Mechanical application of the formula: Without conceptual understanding, students memorise $S = \frac{a}{1-r}$ but cannot explain why it works or when it is valid (e.g., the condition $|\, r \,| < 1$).

- Inability to generalize: Many students can handle a specific numeric series (e.g., $\frac{1}{4} + \frac{1}{16} + \cdots$) but struggle to abstract the general formula for symbolic $a$ and $r$.
- Lack of visual intuition: Purely symbolic instruction often fails to provide an intuitive anchor for why the sum converges, leaving students with only rote procedures.

Together, these theoretical and empirical difficulties suggest a strong need for instructional approaches that bridge the gap between finite operations and infinite totality—a gap that visual models, such as the area model, may help to close.

### *Related research on visual representations for infinite processes*

Prior studies have highlighted the value of visual models for teaching concepts of infinity. For example, Ely (2011) showed that projecting finite properties (such as boundedness or symmetry) onto infinite processes helps learners envision actual infinity. Fischbein (2001) argued that tacit models (implicit visual or physical intuitions) can either support or hinder the understanding of infinity; well-chosen models can serve as bridges across the potential-actual gap.

More specifically, Villabona et al. (2024) demonstrated that acting on totalities of infinite processes (geometric series, repeating decimals) requires constructing multiple "facets" through different actions on the totality. Visual representations provide a rich ground for such actions—learners can measure, compare, and predict, each time building a new facet of the object conception. In contrast, purely symbolic treatments often limit students to a single facet (algebraic manipulation), leaving the object conception fragile.

The present study extends this line of research by proposing a concrete, low-cost area model (iterative square subdivision) that is particularly suited for introducing infinite geometric series at the high school level, prior to formal limit instruction.

## METHODOLOGY

### *Research design*

This study employs a didactic engineering approach (Artigue, 2014), which is considered particularly suitable for designing, implementing, and analyzing teaching sequences based on a precise theoretical framework. Our design follows the APOS theory and focuses on the genetic decomposition (GD) of the concept "sum of an infinite geometric series." The GD hypothesizes the cognitive stages (Action, Process, Object, Schema) that learners are expected to go through during the activity. The engineering process consists of four phases: (1) preliminary analysis, (2) design and a priori analysis, (3) experimentation, and (4) a posteriori analysis. The didactic engineering framework does not prescribe a single type of data analysis. Accordingly, we collected both quantitative and qualitative data. Quantitative data (percentages of students at each APOS level) were derived from the worksheets. Qualitative data (students' written justifications and interview transcripts) were analyzed to identify the cognitive mechanisms (interiorization, completion, encapsulation) and to illustrate typical responses for each APOS level. This mixed-approach is consistent with prior DE studies that combine a priori predictions with a posteriori comparisons (Artigue, 2014).

### *Context and participants*

The study was conducted with Grade 11 who have already learned finite geometric sequences and series but have not yet been formally introduced to the concept of infinite geometric series and limits. The activity is designed as a 90-minute classroom session (or two 45-minute periods). Participants were recruited from intact classes in a Vietnamese upper secondary school. The sample size is 52 students, allowing for qualitative analysis of written responses and follow-up interviews.

### *Description of the teaching activity*

While Nam et al. (2024) used an equilateral triangle model in PowerPoint, we chose a unit square for two reasons. First, the square's right angles and equal sides make the fractional area calculations ($\frac{1}{4}, \frac{1}{16}, ...$) more transparent for students. Second, the square's bounded outline visually reinforces the concept of 'totality' more directly than a triangle, as students often perceive a square as a complete unit. This design choice aligns with our APOS-based focus on the completion mechanism.

The activity follows the four questions described in the following. Each question is designed to trigger specific APOS levels.

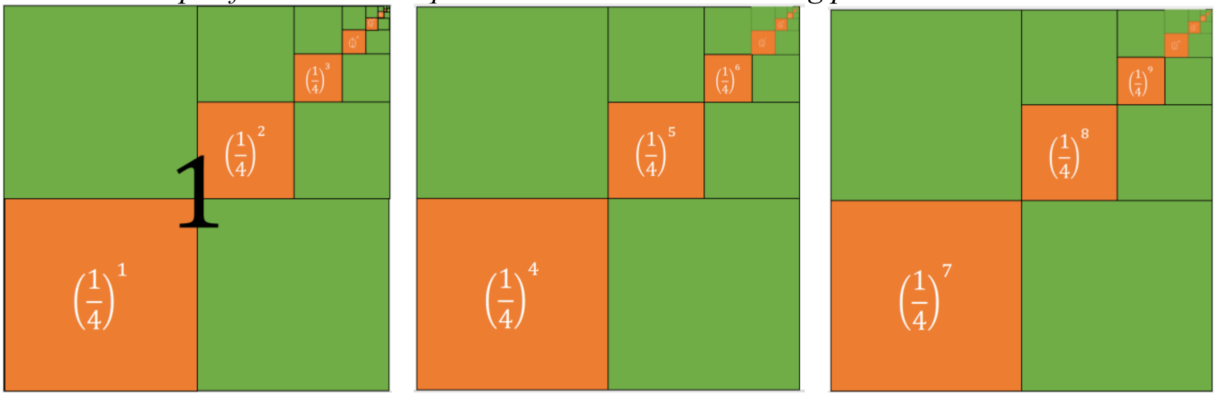
Context: The unit square below is divided and colored according to the following rule:

Step 1: Divide the square into 4 equal parts, color 1 part.

Step 2: Choose one of the 3 uncolored parts (as shown in the figure), divide it into 4 equal parts, color 1 part of that part according to the same rule.

Repeat this process infinitely many times.

**Figure 1.**
*First three steps of the iterative square subdivision and coloring process*




*Note*. The Authors.

Question 1 (Action → Process)

- Task: Compute the colored areas step by step: $u_1, u_2, u_3, \ldots$ and write the general term.
- APOS target:
  - *Action:* Students calculate $u_1 = \frac{1}{4}, u_2 = \frac{1}{16}, u_3 = \frac{1}{64}$ by direct measurement or fraction multiplication.
  - *Process:* Through the hint ("each term equals the previous term multiplied by what number?"), students interiorize the multiplicative rule $u_n = u_{n-1} \times \frac{1}{4}$ and express $u_n = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^{n-1}$.
- Materials: The unit square diagram showing the first three steps of subdivision and coloring (three images provided).

Question 2 (Process → Object / Totality)

- Task: Observe the images and predict the total-colored area after infinitely many steps.
- APOS target: Students move from the potential infinity (continuing the process) to actual infinity (the completed totality). The boundedness of the unit square (total area = 1) visually suggests that the infinite sum cannot exceed 1, and the repetitive

pattern leads to an intuitive estimate (e.g., $\frac{1}{3}$). This is the first step toward completion and encapsulation.

- Materials: Same images, possibly with an additional prompt: "What fraction of the whole square do you think will finally be colored?"

Question 3 (Object → Formal Proof)

- Task: Using the formula for the sum of the first $n$ terms of a geometric series and the concept of limit, prove that $S = \frac{\frac{1}{4}}{1-\frac{1}{4}}$.
- APOS target: Students confirm the intuitive estimate (from Q2) by a rigorous algebraic limit argument. They apply the closed-form formula $S_n = u_1 \frac{1-q^n}{1-q}$ and compute $\lim_{n\to+\infty} S_n = \frac{u_1}{1-q}$. At this stage, the infinite sum becomes an Object – a static mathematical entity that can be manipulated without referring to the infinite process. Students also see that the result matches the visual prediction, reinforcing encapsulation.

Question 4 (Schema – Generalization)

- Task: State the result for the general case (first term $a$, common ratio $r$, with $0 < r < 1$) and, if possible, draw a corresponding geometric model.
- APOS target: Students extend the specific case to a general formula $S = \frac{a}{1-r}$, showing that they have constructed a Schema connecting area interpretation, geometric series, and limits. They may also recognize conditions for convergence ($| r |< 1$).

### *Data collection instruments*

Three types of data were collected:

1. Student worksheets: Each student completes the worksheet individually (or in pairs) containing the four questions. The worksheets collect written explanations, calculations, and predictions.
2. Semi-structured interviews: After the activity, 6–8 students (selected based on varied performance) were interviewed to probe their reasoning in depth. Interview questions focus on: How they recognized the pattern; How they interpreted

"infinitely many steps"; Whether they see the formula as a "completed object"; Connection between the square diagram and the limit process.

3. Classroom observations: The researcher takes field notes on student discussions, difficulties, and spontaneous use of visual reasoning.

### *Data analysis: Quantitative and qualitative integration within the didactic engineering framework*

The didactic engineering framework (Artigue, 2014) does not impose a single type of data analysis; rather, the a posteriori analysis phase is tailored to the theoretical constructs and research questions. Since our study is guided by APOS theory and aims to identify both the distribution of mental structures (Action, Process, Object, Schema) and the underlying cognitive mechanisms (interiorization, completion, encapsulation), we employed a mixed-approach combining quantitative and qualitative analyses. This integration is consistent with prior DE studies that compare a priori predictions with observed outcomes (Villabona et al., 2024).

### *Quantitative analysis*

The primary quantitative data source was the student worksheets (N = 52). Each student's responses to the four questions were coded according to the highest APOS level demonstrated, using the operational indicators defined in Table 1.

Two researchers independently coded 20% of the worksheets (randomly selected) using the operational indicators in Table 1. Inter-rater agreement reached 86%; disagreements were resolved through discussion. The remaining worksheets were coded by the first author.

For each question, we calculated the percentage of students at each APOS level (Action, Process, Object, Schema). These frequencies were tabulated to provide a global overview of student performance.

**Table 1.**
*Operational indicators for coding APOS levels*

| APOS Level | Operational indicators in this activity |
|---|---|
| Action | Computes $u_1, u_2, u_3$ correctly but cannot identify the general term without step-by-step calculation; needs external prompting to continue. |

| APOS Level | Operational indicators in this activity |
|---|---|
| Process | Automatically writes $u_n = (\frac{1}{4})^n$ or $u_n = u_{n-1} \times \frac{1}{4}$; can extend the pattern mentally without calculating each term; describes the process as “repeating forever”. |
| Object | Uses the closed-form formula $S = \frac{a}{1-r}$ appropriately; justifies $S = \frac{1}{3}$ via limit of partial sums or via the visual totality “the coloured region is exactly $\frac{1}{3}$ of the square”; can apply the formula to new contexts (e.g., different $a, r$). |
| Schema | Connects the area model, algebraic series, and limit concept; explains why (r<1) is necessary; can construct a similar visual model for another ratio (e.g., $r = \frac{1}{3}$ by dividing into 3 parts). |

*Note*. The Authors.

For each question, we calculated:

- The percentage of students at each APOS level (Action, Process, Object, Schema).
- The percentage of students who correctly predicted the limit in Q2 and who successfully generalized in Q4.

These frequencies were tabulated (Table 3 in Results) to provide a global overview of student performance and to test the a priori hypotheses derived from the genetic decomposition.

### *Qualitative analysis*

The qualitative analysis aimed to understand how students progressed (or failed to progress) through the APOS levels and to identify the activation of completion and encapsulation mechanisms. Data sources included:

- Written justifications on the worksheets (e.g., explanations of the pattern, predictions, and proofs).
- Semi-structured interview transcripts (8 students, selected for variation in performance).

We employed thematic coding following the analytical procedures outlined by Miles et al. (2014). The coding framework was theoretically grounded in APOS theory (Arnon et al., 2014) to capture students’ cognitive mechanisms.

**Table 2.**
*The coding scheme included*

| Code | Definition |
|---|---|
| *Action indicator* | Step-by-step calculation; needs external prompting; no generalization. |
| *Process indicator* | Recognizes multiplicative rule; can state general term without computing each step; describes "never-ending" process. |
| *Completion indicator* | Envisions the infinite process as a finished whole ("the colored area will eventually be $\frac{1}{3}$"). |
| *Encapsulation indicator* | Treats the infinite sum as a static object; uses closed-form formula as a tool. |
| *Schema indicator* | Connects visual, algebraic, and limit representations; generalizes to arbitrary $a, r$. |

*Note*. The Authors.

The number of interviewees (n = 8) was determined by data saturation, the point at which additional interviews no longer yielded new themes or categories related to the APOS mechanisms. This is consistent with common practice in qualitative research within didactic engineering (Artigue, 2014).

Interview transcripts were analyzed in full. Representative verbatim quotes were selected to illustrate typical responses at each level, as presented in the Results section. The qualitative analysis also served to validate or refine the quantitative coding—for example, a student might have written a correct formula (coded as Object) but in the interview revealed a pure process-based understanding; in such cases we triangulated and assigned the lower level.

### Integration within the a posteriori analysis phase of DE

Following the didactic engineering model, the a posteriori analysis consisted of comparing the observed outcomes (both quantitative distributions and qualitative patterns) with the a priori predictions (the genetic decomposition). This comparison allowed us to:

- Confirm or disconfirm the hypothesized cognitive pathways.
- Identify unexpected findings (e.g., early completion in Q2).
- Detect specific difficulties (e.g., low Schema attainment in Q4) that indicate needed adjustments in the teaching design.

Thus, quantitative data provided a macro-level picture of overall effectiveness, while qualitative data provided a micro-level understanding of cognitive mechanisms. The two forms of analysis are complementary and together support the iterative refinement characteristic of didactic engineering.

## ETHICAL CONSIDERATIONS

Informed consent was obtained from school authorities, parents, and students. Participation is voluntary. Data is anonymized (student codes instead of names). The study follows the ethical guidelines for educational research.

## RESULTS

This section presents the outcomes of the didactic engineering experiment. We first summarize the a priori analysis (expected APOS levels for each question) and then report the actual students' performance based on worksheets (n = 52) and follow-up interviews (n = 8). The results are organized around the two research questions.

### *Overview of student performance on the four tasks*

All 52 students completed the worksheet. Table 3 shows the percentage of students who demonstrated each APOS level in the final answer for each question (coding based on the operational indicators defined before).

**Table 3.**
*Distribution of APOS levels across the four tasks (n = 52)*

| Task | Action only | Process | Object | Schema |
|---|---|---|---|---|
| Q1 (terms & general term) | 13.5% | 75.0% | 11.5% | 0% |
| Q2 (predict total area) | 9.6% | 48.1% | 36.5% | 5.8% |
| Q3 (prove using limit) | 0% | 11.5% | 61.5% | 26.9% |
| Q4 (general case) | 7.7% | 15.4% | 42.3% | 34.6% |

*Note*. The Authors. A student can be classified at the highest level observed in the justifications "Action only" means they gave concrete calculations without any generalization.

## ANSWERING RQ1: MENTAL STRUCTURES AND MECHANISMS FOR OPERATING ON INFINITE ENTITIES

RQ1: *According to APOS theory, what mental structures (Action, Process, Object, Schema) and cognitive mechanisms (interiorization, encapsulation, completion) enable students to successfully operate on infinite entities when learning the sum of an infinite geometric series?*

The analysis of student responses and interviews suggests progression through the APOS levels, consistent with the genetic decomposition.

### *Action level*

About 13–14% of students remained at the Action level in Q1 and Q2. For example, student S17 wrote:

"$u_1 = \frac{1}{4}$, $u_2 = \frac{1}{16}$, $u_3 = \frac{1}{64}$. I don't see a rule, I just draw each step and count."

These students could compute the first three terms by following the diagram but could not state the general term without external prompting. They treated each coloring as a separate event, not as an iterative process. In Q2, they either gave no prediction or guessed "I think it will be 1 because we color many times". They still operated on "potential infinity"— an endless continuation without closure.

### *Process level*

Most students (75% in Q1, 48% in Q2) reached the Process level. They interiorized the multiplicative pattern:

S24: "Each time we take one uncolored quarter and color one of its four parts, so the new colored area is $\frac{1}{4}$ of the previous colored area. So $u_n = (\frac{1}{4})^n$."

These students could mentally run the repetition without recalculating each step. They described the process as "going on forever", but still struggled to treat the *infinite sum* as a completed whole. In Q2, they often answered "It will be close to $\frac{1}{3}$ but never reach it"—a typical "potential infinity" view.

### Object level and the completion mechanism

A substantial proportion (36.5% in Q2, 61.5% in Q3) demonstrated an Object conception, especially after using the limit formal proof. The crucial mechanism was completion— the shift from "never ending" to "already finished".

Interview excerpt (S31, after Q2 and before Q3):

Interviewer: "You guessed the total-colored area is $\frac{1}{3}$. But how can you add an infinite number of pieces?"

S31: "Because the whole square is only area 1. The colored parts cannot exceed that. And I saw that after each step, the uncolored part is always three times the colored part we just added. So, in the end, the colored must be $\frac{1}{4}$ of the total uncolored? Wait… Actually, if the uncolored is three times the colored, then colored + 3×coloured = 1, so colored = $\frac{1}{4}$? No that's wrong… Let me think: every time we colour a new tiny square, the leftover is still three times that new piece. So, in the limit, the total colored area S satisfies S = $\frac{1}{4}+(\frac{1}{4})^2 + \ldots$ and also the remaining white area is three times S? Because at each stage white = 3 × (what we have colored so far)? Actually at step 1: white=$\frac{3}{4}$, coloured =$\frac{1}{4}$, white=3×coloured. Step 2: colored $=\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$, white $=\frac{11}{16}$, not exactly 3×coloured. But by using the formula I trust it."

This appears to show a partial object – the student attempts to encapsulate but still hesitates. However, after Q3 (limit proof), the same student wrote:

"We have $S_n = \frac{1}{4}.\ \frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}}$. As $n\to\infty$, $(\frac{1}{4})^n \to 0$, so $S_n \to \frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$. So, the infinite sum is exactly $\frac{1}{3}$, not just an approximation."

Here the infinite sum is treated as a static object (the limit value) by this student. The encapsulation of the limit process into the closed form $\frac{a}{1-r}$ was observed in 61.5% of students for Q3.

### Schema level

In Q4 (general case), 34.6% of students constructed a Schema: they correctly wrote $S = \frac{a}{1-r}$ for $0 < r < 1$, and some even produced a geometric representation (e.g., dividing a unit segment or square into $\frac{1}{r}$ parts). Student S44 wrote:

"If we start with a square of area a and each time we take a fraction r of the previous piece, the total area is a + ar + ar² + … = $\frac{a}{1-r}$. The same works if we draw a rectangle of area a and keep subdividing the leftover by ratio r."

This student connected the algebraic formula, the convergence condition, and the visual model—a clear Schema.

Answer to RQ1: The successful operation on infinite entities (sum of infinite geometric series) requires:

- Interiorization of the iterative multiplication pattern (Action → Process);
- Completion (envisioning the infinite process as a finished totality) facilitated by the bounded visual model;
- Encapsulation of the process into the static closed-form formula (Process → Object);
- Thematic connection between geometric, algebraic, and limit representations (Schema).

Students who remained at Action or Process levels failed to accept "actual infinity" and often rejected the idea that an infinite sum can have a finite exact value.

## Answering RQ2: Impact of the visual area model on encapsulation and completion

RQ2: *How does the use of a concrete visual area model (iterative unit square subdivision) affect students' encapsulation of an infinite process and their completion mechanism, helping them achieve a Totality and Object conception?*

The data suggest that the visual model played an important role, especially in triggering the completion mechanism and providing intuitive anchors for encapsulation.

### *Visual model as a support for completion*

In Q2 (prediction before formal proof), 36.5% of students already gave the correct answer $\frac{1}{3}$, and another 48.1% gave a number close to $\frac{1}{3}$ (e.g., 0.33, 0.3). When asked how they obtained the prediction, most referred to the diagram:

S12: "Look at the three pictures. The first one: $\frac{1}{4}$ coloured. Second: $\frac{1}{4}+\frac{1}{16}=\frac{5}{16}=0.3125$. Third: $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}=\frac{21}{64}\approx 0.328$. It's getting closer to $\frac{1}{3}$. And the square is limited, so it can't pass $\frac{1}{3}$."

Even though they had not yet learned the limit theorem, the *boundedness* and *visual convergence* of the colored region suggested that the process "ends" at a fixed portion of the square. This intuitive completion allowed students to treat the infinite process as a finished totality before performing algebraic proof.

### *Visual model as an anchor for encapsulation*

In the interviews, students who struggled with the algebraic limit (e.g., S07) could still articulate the result via the visual totality:

"I don't really understand the limit thing, but I see that after many steps the white parts become tiny and the colored area stops changing. It looks like exactly one third of the square is colored—you can see from the pattern that every time the new colored piece is three times smaller than the remaining white, so in the end the colored is $\frac{1}{3}$."

This student appeared to encapsulate the infinite process into a static visual object (the final colored proportion) without fully formalizing the limit. For them, the area model acted as a bridge to the Object level. After instructions, they could apply the formula to other problems (e.g., $a = 2, r = \frac{1}{2} \rightarrow$ S=4) by analogy, indicating an operational Object.

### *Comparison with a hypothetical purely algebraic instruction*

While not part of this study's design, we asked six students during retrospective interviews about their previous experience with infinite geometric series (they had none). When presented with the same questions but only the algebraic definition ($u_n = ar^{n-1}$), they said they would have simply memorised the formula. The visual model made them "see why the formula makes sense" (S29). This suggests that the area model significantly lowers the barrier to completion and encapsulation by providing a concrete, bounded totality.

Answer to RQ2: The visual area model positively impacts encapsulation and completion by:

- Making the bounded totality (unit square) perceptually available, which supports the completion of an otherwise endless process.

- Providing an intuitive visual anchor for the limit value, allowing students to *see* the sum as a static portion of the whole.
- Helping students who struggle with formal limit manipulations to still achieve an Object conception (via visual encapsulation), thereby scaffolding the transition to full Schema.

### *A posteriori comparison with a priori analysis (didactic engineering)*

The a priori analysis predicted that most students would reach Process level by Q1 and Object level by Q3. Table 1 confirms this. However, a few deviations were noted:

- Unexpected difficulty in generalization (Q4): 34.6% reached Schema, but 15.4% remained at Process, and 7.7% at Action. These students could not transfer the specific ratio $r = \frac{1}{4}$ to arbitrary $r$. They wrote $S = \frac{a}{1-\frac{1}{4}}$ instead of $\frac{a}{1-r}$. This indicates that the Schema was not fully constructed for all.
- Early completion in Q2: More students than expected (36.5%) gave the exact $\frac{1}{3}$ prediction, often using a "ratio of leftovers" argument (e.g., total white = 3 × total colored). This suggests that the visual model may have allowed some students to bypass the algebraic limit and generate a "visual proof", which is a positive finding.

These observations will inform a future iteration of the didactic engineering (e.g., adding a bridging task for general $r$).

## DISCUSSION

This section interprets the findings in relation to the two research questions. We first answer RQ1 by examining the mental structures and mechanisms students used when operating on the infinite series. Then we address RQ2 by evaluating the specific contribution of the visual area model to the encapsulation and completion processes. Finally, we discuss unexpected findings and limitations.

### *Answering RQ1: Mental structures and mechanisms for operating on infinite entities*

RQ1 asked: *What mental structures (Action, Process, Object, Schema) and cognitive mechanisms (interiorization, encapsulation, completion) enable students to successfully operate on infinite entities when learning the sum of an infinite geometric series?*

Our results confirm that students who successfully reached the Object or Schema level employed three key mechanisms in sequence:

- Interiorization (Action → Process): By computing the first three colored areas ($u_1 = \frac{1}{4}, u_2 = \frac{1}{16}, u_3 = \frac{1}{64}$) and noticing the repeated multiplication by $\frac{1}{4}$, students internalised the pattern. This was observed in 75% of students in Q1, who could state the general term $u_n = (\frac{1}{4})^n$ without step-by-step calculation.
- Completion (Process → Totality): Students who envisioned the endless subdivision as a finished whole (e.g., "the colored area will stop changing at $\frac{1}{3}$") demonstrated the completion mechanism. This required shifting from potential infinity ("it never ends") to actual infinity ("it is already a fixed portion of the square"). In Q2, 36.5% of students predicted exactly $\frac{1}{3}$, and another 48.1% gave a close value, indicating that completion had occurred for a substantial proportion even before formal limit instruction.
- Encapsulation (Process/Totality → Object): Students who encapsulated the infinite sum turned the dynamic process into a static object—the closed-form formula $S = \frac{u_1}{1-q}$. In Q3, 61.5% of students correctly applied the limit of partial sums to prove that $S = \frac{1}{3}$, and they could subsequently use the formula to solve new problems (e.g., in Q4, 42.3% reached Object level for the general case). These students treated the infinite sum as a *Transcendent Object* (Brown et al., 2010)—a finite value derived from an infinite process, which does not inherit all properties of partial sums.

Students who remained at Action or Process levels (e.g., 13.5% in Q1, 9.6% in Q2) exhibited typical potential-infinity difficulties: they could not state the general term without external cues, or they insisted that the sum "approaches but never reaches" $\frac{1}{3}$, reflecting an inability to complete the process.

Thus, the answer to RQ1 is that successful operation on the infinite geometric series requires interiorization, completion, and encapsulation in sequence, and the absence of any one mechanism results in a fragmented (Action/Process) conception. Our findings extend Villabona et al. (2024) by showing that completion can be triggered earlier (even before formal limit instruction) when a bounded visual model is used, and that encapsulation of the infinite sum does not necessarily require mastery of algebraic limits—the visual totality alone can serve as an anchor for some students.

### *Answering RQ2: Impact of the visual area model on encapsulation and completion*

RQ2 asked: *How does the use of a concrete visual area model (iterative unit square subdivision) affect students' encapsulation of an infinite process and their completion mechanism, helping them achieve a Totality and Object conception?*

The visual model appeared to be important in three ways:

- Providing a bounded totality: Unlike purely symbolic series, the unit square has a perceptible boundary (total area = 1). This physical constraint gave students a concrete referent for "completion". In interviews, students repeatedly referred to the fact that "the colored parts cannot exceed the whole square" as the reason they could imagine a final value. This finding aligns with Fischbein (2001), who argued that tacit models of boundedness scaffold the shift to actual infinity. The role of a bounded visual model in triggering completion is consistent with the findings of Nam et al. (2024), who used a dynamic triangle model in PowerPoint to illustrate the same series. Their students also visually inferred the limit 1/3 by observing that the white central triangle could be partitioned into three equal parts. However, our study extends this observation by showing that even static images of the first three subdivision steps (without animation) were sufficient for 36.5% of students to predict the exact limit, suggesting that the perceptual boundedness of the unit square—not necessarily motion effects—is the critical factor for completion.
- Supporting completion through visual convergence: The three images (steps 1–3) clearly showed that the newly added area became smaller each time and that the colored region approached a fixed portion of the square. Even without algebraic limits, 36.5% of students predicted the exact limit 1/3 (and many more a close

value). This suggests that the visual pattern alone can might trigger the completion mechanism, reducing the need for abstract limit reasoning at the initial stage.

- Facilitating encapsulation of the infinite sum: Once students had visualized the totality (the colored region as a finished whole), they more readily accepted that the infinite arithmetic sum $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$ equals a single number ($\frac{1}{3}$). In Q3, 61.5% of students successfully performed the algebraic proof and used the formula as an object. Notably, even students who struggled with formal limits (e.g., S07 in the interviews) could still describe the sum as a static object by referring to the visual totality: "I see that after many steps, the colored area is exactly 1/3 of the square." This demonstrates that the area model can serve as a stand-alone anchor for encapsulation when symbolic manipulation is not yet mastered.

In comparison with purely algebraic instruction (as hypothesized from prior literature), the visual model lowered the cognitive barrier for both completion and encapsulation, especially for students with weaker procedural skills. However, the model alone did not automatically lead to Schema (only 34.6% reached that level), indicating that additional bridging activities are needed for full generalization.

Thus, the answer to RQ2 is that the visual area model positively affects completion by providing a perceptually bounded totality and visual convergence, and it supports encapsulation by giving a concrete referent for the static infinite sum, thereby making the Transcendent Object more accessible.

### *Comparison with prior theoretical frameworks and empirical studies*

Fischbein (2001) argued that intuitive models of infinity (tacit models) can be either supportive or obstructive. Students often possess an implicit model of infinity as an unending, dynamic process—a potential infinity—which blocks acceptance of actual infinity. Our visual area model functions as a supportive tacit model by projecting boundedness—a finite, perceptible property—onto the infinite process. The unit square's clear outline and fixed total area (1) provide a concrete "container". This finding is empirically consistent with Fischbein's hypothesis that a well-chosen visual anchor can convert a potential obstacle (the endlessness) into a facilitator (the bounded totality). In our study, 36.5% of students correctly predicted the limit $\frac{1}{3}$ before any formal instruction, often explicitly referring to the square's limited area.

Sierpinska (1987) documented that students often treat limits as unattainable approximations because they project properties of finite partial sums onto the infinite limit. Ely (2011) later proposed that "envisioning the infinite by projecting finite properties" could be productive if the projected property is appropriately chosen. Our study provides a concrete instance of productive projection: the boundedness of the square (a finite property) is projected onto the infinite sum, leading to acceptance of a finite limit. Moreover, we observed that students who remained at the Process level (e.g., those who said "the sum approaches but never reaches $\frac{1}{3}$") exhibited similar to the epistemological obstacle described by Sierpinska. In contrast, students who reached the Object level successfully distinguished between partial sums and the complete limit, overcoming that obstacle.

Nam et al. (2024) used a dynamic triangle model in PowerPoint to illustrate the same series ($q = \frac{1}{4}$). Their action research showed that the model enhanced engagement and helped students visually infer the sum $\frac{1}{3}$. However, our APOS-based analysis suggests additional insights not explicitly addressed by their reification/procept framework. Specifically, we found that:

- Completion (envisioning the finished totality) may be triggered by static images alone (three subdivision steps), not necessarily by dynamic zoom effects.
- Encapsulation occurred either via the algebraic limit (61.5% of students) or via the visual totality alone – a finding not reported by Nam et al. (2024).
- Generalization to arbitrary ratios ($S = \frac{a}{1-r}$) proved difficult (only 34.6% reached Schema), a limitation not addressed in their study because they also used only $q = \frac{1}{4}$.

Methodologically, our didactic engineering approach with a priori genetic decomposition provides a stronger basis for comparing predicted and observed cognitive pathways than action research.

Artigue (1998) highlighted the difficulty students face when transitioning from the algebraic "finite" thinking to the analytic "infinite" thinking, particularly with limits. Our study operationalizes this transition through the APOS mechanisms of completion and encapsulation. While previous APOS studies on limits (e.g., Arnon et al., 2014) focused on functions, our work extends the framework to infinite geometric series, showing that the same mechanisms apply.

Moreover, we provide empirical evidence that a bounded visual model can serve as a "didactic cut" (Artigue, 1998)—a carefully designed intervention that helps students leap over the epistemological obstacle of actual infinity.

### *Unexpected findings and didactic implications*

Two unexpected outcomes merit attention beyond the two RQs. First, the high accuracy of early predictions in Q2 suggests that the model may offer a *quasi-proof* by visual pattern. Teachers could leverage students' intuitive estimates as a starting point for the formal limit argument, creating a cognitive need for algebraic proof. Second, the modest Schema attainment (34.6%) indicates that generalization to an arbitrary ratio $r$ remains challenging, consistent with APOS theory's claim that Schema construction is the most demanding stage (Arnon et al., 2014). To strengthen transfer, we recommend adding bridging tasks—for example, asking students to construct a similar square subdivision for $r = \frac{1}{3}$ or $r = \frac{2}{3}$ before presenting the general formula $S = \frac{a}{1-r}$. The difficulty in generalizing to arbitrary ratios (only 34.6% reached Schema) may be partly attributed to the specificity of the ratio $\mathrm{q} = \frac{1}{4}$, which has the convenient property that the remaining area is always three times the newly colored part—a pattern that can be exploited for a visual shortcut. A similar limitation appears in the study by Nam et al. (2024), where the triangular model also used $\mathrm{q} = \frac{1}{4}$ and did not explicitly address generalization to other ratios. This suggests that a visual model tailored to a single ratio, while effective for initial concept formation, may inadvertently hinder abstraction unless followed by deliberate bridging activities (e.g., constructing a model for $\mathrm{q} = \frac{1}{3}$ or $\mathrm{q} = \frac{2}{3}$).

### *Pedagogical implications*

Based on our findings, we offer the following concrete recommendations for teaching the sum of an infinite geometric series:

- Present the visual model before the algebraic formula. Allow students to predict the limit intuitively by observing the first three iterations of the colored square. This creates a cognitive need for the formal proof and makes the formula appear as a natural confirmation rather than a magical rule.
- Use the visual totality as an alternative path to encapsulation. For students who struggle with algebraic limits or limit notation, teachers can ask: "What fraction of

the whole square do you think is finally colored?" without requiring limit symbols. Visual boundedness alone can support object conception.

- Address the difficulty in generalization explicitly. Because only 34.6% of students spontaneously generalized to $S = \frac{a}{1-r}$, teachers should add bridging tasks before introducing the symbolic formula. For example:
  - Ask students to repeat the activity with a different ratio (e.g., $q = \frac{1}{3}$ by dividing a triangle or a rectangle into three parts).
  - Guide students to replace the concrete number "$\frac{1}{4}$" with a variable $r$ in their written explanation.
  - Compare two visual models (e.g., square with $q = \frac{1}{4}$ and triangle with $q = \frac{1}{3}$) to highlight the invariant structure.
- Use the model to confront potential infinity misconceptions. When a student says "the sum approaches but never reaches $\frac{1}{3}$", ask: "If the process never ends, can we still point to a fixed region of the square that is coloured?" This question directs attention to the completed totality.
- Integrate the visual activity with the formal limit proof. After students have predicted $\frac{1}{3}$ from the square, present the partial sum formula and limit derivation as a way to confirm their intuition, not as an entirely new idea.

## Limitations and future research

This study has several limitations that should be addressed in future research:

- Single ratio and model. We used only $q = \frac{1}{4}$ and a unit square. It is unclear whether the same completion and encapsulation mechanisms would be observed with other ratios (e.g., $q = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$) or other geometric shapes (e.g., rectangle, line segment). Future studies should systematically vary the ratio and the visual configuration.
- Static images vs. dynamic models. Our activity used three static images (printed or displayed). While this sufficed for many students, dynamic models (such as the PowerPoint zoom effect used by Nam et al., 2024) might enhance the sense of "infinite continuation" for some learners. A comparative study could examine

whether adding animation strengthens completion, especially for students with weaker spatial visualization.

- No long-term retention measure. We assessed students' understanding immediately after the activity. Longitudinal studies are needed to determine whether the Object conception remains stable after several weeks or months, or whether students revert to process-based thinking without reinforcement.
- Lack of a control group. We did not compare our visual-model instruction with a purely algebraic instruction group. A randomized controlled experiment would provide stronger causal evidence for the effectiveness of the area model.
- Sample homogeneity. The study was conducted in a single Vietnamese high school. Replication in different cultural and educational contexts is necessary to assess generalizability.
-

Future research directions:

- Design and test a version of the activity for additional ratios (e.g., $q = \frac{1}{3}$ using a triangle subdivided into three parts).
- Combine the APOS-based worksheet with dynamic software (GeoGebra, PowerPoint) and compare outcomes.
- Investigate whether early success with the visual model transfers to other infinite processes (repeating decimals, Zeno's paradoxes).
- Conduct a teaching experiment that explicitly bridges from the specific ratio to the general formula, measuring Schema development over several lessons.

## FINAL CONSIDERATION

This study set out to design and evaluate an APOS-based teaching activity that uses an iterative unit square subdivision to introduce the sum of an infinite geometric series. The analysis indicates that students' understanding develops through a sequence of cognitive movements: first, the interiorization of the multiplicative pattern; then, the completion of the infinite process as a bounded totality; and finally, the encapsulation of that process into a stable mathematical object. In this sense, the visual area model did not merely illustrate a formula already known by students, but helped create conditions for the formula to become meaningful.

The results show that the visual model supported students in moving from a procedural view of repeated additions to a more integrated understanding of the infinite sum. The bounded square offered a concrete reference for the idea of totality, allowing many students to anticipate the limiting value before formal algebraic treatment. At the same time, the findings also indicate that visual intuition alone is not sufficient for all dimensions of conceptual development. Generalizing from the specific visual model to arbitrary ratios remained a demanding step, which shows the need for additional tasks that explicitly connect the concrete representation, the algebraic structure and the convergence condition.

The main contribution of the study lies in the articulation between didactic design and cognitive analysis. The proposed worksheet offers a low-cost and easily applicable classroom resource, while the APOS-based analysis clarifies why and under what conditions the visual model can support learning. The study also refines the genetic decomposition of infinite geometric series by showing that completion and encapsulation may emerge through visual reasoning before full mastery of formal limit notation.

These results suggest that teachers should introduce infinite geometric series through carefully structured visual experiences before presenting the algebraic formula as a finished rule. The model should be followed by guided generalization activities, comparisons with other ratios and explicit discussion of the difference between approaching a value and accepting the infinite sum as an exact mathematical object. In this way, the activity can contribute not only to procedural performance, but also to a deeper understanding of infinity, limit and convergence in school mathematics.

The study is limited by its use of a single ratio, static images and one school context. Future investigations may compare different visual configurations, include dynamic digital models, analyze long-term retention and contrast the proposed design with purely algebraic instruction. Such developments would make it possible to evaluate more precisely the scope of the model and its potential for broader use in mathematics education.

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## *CRediT Author Statement*


- **Acknowledgments**: The authors are grateful to the respondents who willingly took part in the study.
- **Funding**: This research was funded by Saigon University.
- **Conflicts of interest**: Authors declare that they have no conflicts of interest.
- **Ethical approval**: Informed consent were obtained from school authorities, parents, and students. Participation was voluntary. Data was anonymized (student codes instead of names). The study follows the ethical guidelines for educational research.
- **Data and materials availability:** The data supporting the findings of this study are available from the authors upon request.
- **Author contributions**: All authors participated equally in the entire process of writing the article.


**Processing and editing: Editora Ibero-Americana de Educação**
Proofreading, formatting, normalization and translation